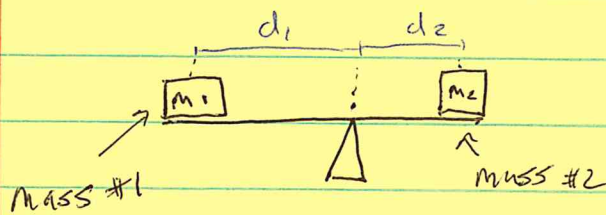


12.4 Applications

Note: Students should review Section 7.6

Center of Mass

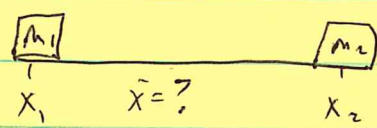


balanced $\Leftrightarrow m_1 d_1 = m_2 d_2$.

Law of the Lever

Archimedes ~ 200 B.C.

Question:

Suppose we have . How do we find the balance point \bar{x} ?

Solution

Let $d_1 = \bar{x} - x_1$, $d_2 = x_2 - \bar{x}$.

Then $m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}) \Rightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$.

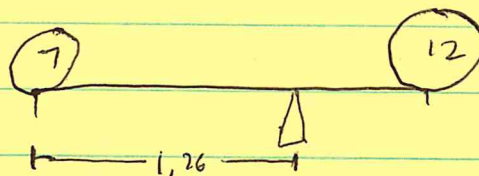
Ex

Given a thin strong board of length ~~10~~ 2 meters with a 7 kg mass on one end and a 12 kg mass on the other, where is the balance pt?

Sol

Let $m_1 = 7$, $x_1 = 0$, $m_2 = 12$, $x_2 = 2$. Then

$$\bar{x} = \frac{7 \cdot 0 + 12 \cdot 2}{7 + 12} = \frac{24}{19} \approx 1.26 \text{ meters.}$$



Try this at home!

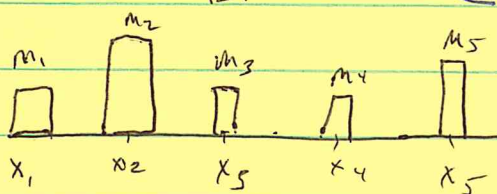
Pf there are several weights then it can be shown that

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n}$$

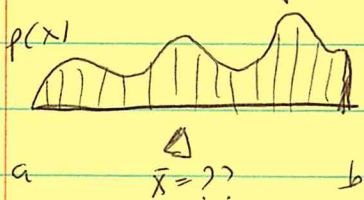
$$= \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}$$

This is called a weighted average.

Total mass.



Now let $p(x)$ be a linear density function: $\frac{\text{mass}}{\text{length}}$



To find the balance point partition $[a, b]$ by $a = x_0 < x_1 < x_2 < \dots < x_n = b$.

Let $M_i = p(x_i) \Delta x$.

Then

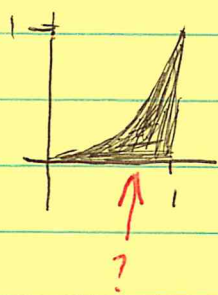
$$\bar{x} \approx \frac{\sum_{i=0}^{n-1} m_i x_i}{\sum_{i=0}^{n-1} m_i} = \frac{\sum_{i=0}^{n-1} p(x_i) x_i \Delta x}{\sum_{i=0}^{n-1} p(x_i) \Delta x}$$

Now, take limit as $\Delta x \rightarrow 0$, $n \rightarrow \infty$, to get

$$\bar{x} = \frac{\int_a^b p(x) x dx}{\int_a^b p(x) dx}$$

Ex Let $p(x) = x^2$ over $[0, 1]$. Find the balance pt. \bar{x} .

Sol

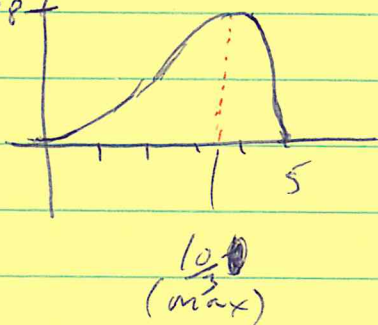


$$\bar{x} = \frac{\int_0^1 x^2 \cdot x \, dx}{\int_0^1 x^2 \, dx} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}.$$

Thus $\bar{x} = \frac{3}{4}$.

Ex Let $p(x) = x^2(5-x)$ over $[0, 5]$. Find \bar{x} .

18.518



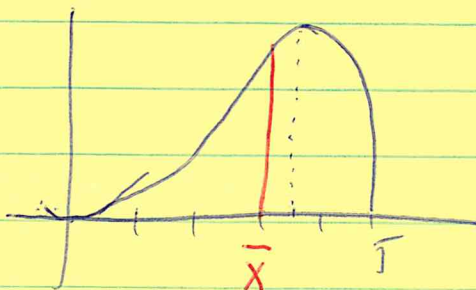
Compare to location of max of $p(x)$.

Sol

To find max: $p'(x) = (5x^2 - x^3)' = 10x - 3x^2$
 $= x(10 - 3x) \stackrel{\text{set}}{=} 0$
 $\Rightarrow x = \frac{10}{3}$

Max is at $x = \frac{10}{3} = 3\frac{1}{3}$, $f(3\frac{1}{3}) = \frac{500}{27} = 18.518$.

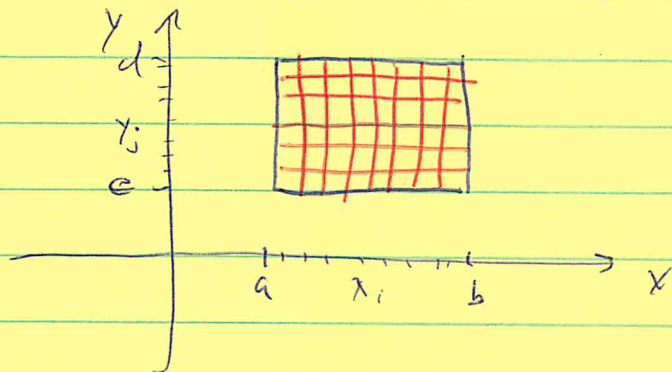
To find \bar{x} : $\bar{x} = \frac{\int_0^5 5x^3 - x^4 \, dx}{\int_0^5 5x^2 - x^3 \, dx} = \dots = 3$.



Now we generalize to two dimensions.

Let $\rho(x,y)$ be a density function ($\frac{\text{mass}}{\text{area}}$) for a rectangle $R = [a,b] \times [c,d]$. We want to find the balance point (\bar{x}, \bar{y}) , also called the center of mass.

We partition R as shown



$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

$$c = y_0 < y_1 < y_2 < \dots < y_m = d$$

$$\Delta x = x_{i+1} - x_i$$

$$\Delta y = y_{j+1} - y_j$$

Let $m_{ij} = \rho(x_i, y_j) \Delta x \Delta y \approx$ mass of cell ij .

Let $M_i = \sum_{j=0}^{m-1} m_{ij} \approx$ mass of column i

Then

$$\bar{x} \approx \frac{\sum_{i=0}^{m-1} M_i x_i}{\sum_{i=0}^{m-1} M_i} = \frac{\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} x_i \rho(x_i, y_j) \Delta x \Delta y}{\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \rho(x_i, y_j) \Delta x \Delta y}$$

Taking limits as $\Delta x, \Delta y \rightarrow 0$ ($m, n \rightarrow \infty$) we get

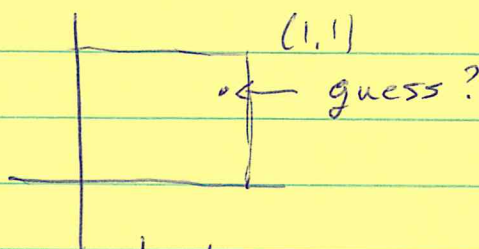
$$\bar{x} = \frac{\int_c^d \int_a^b x \rho(x,y) dx dy}{\int_c^d \int_a^b \rho(x,y) dx dy} = \frac{M_y \leftarrow \text{moment w.r.t } y\text{-axis}}{M \leftarrow \text{total mass}}$$

$$\text{Likewise } \bar{y} = \frac{M_x}{M} = \frac{\iint \cancel{y} \rho(x,y) dx dy}{\iint \rho(x,y) dx dy}.$$

This holds for regions that are not rectangles.

Ex Find the mass and center of mass of the square $[0,1] \times [0,1]$ with density function $\rho(x,y) = x^2 + 2y + 1$

Sol



$$M = \int_0^1 \int_0^1 x^2 + 2y + 1 dx dy = \int_0^1 \left[\frac{1}{3} + 2y + 1 \right] dy = \frac{4}{3} + 1 = \boxed{\frac{7}{3}}$$

$$M_y = \int_0^1 \int_0^1 x(x^2 + 2y + 1) dx dy = \int_0^1 \int_0^1 x^3 + 2xy + x dx dy$$

$$= \int_0^1 \left[\frac{1}{4} + y + \frac{1}{2} \right] dy = \frac{3}{4} + \frac{1}{2} = \boxed{\frac{5}{4}}$$

$$M_x = \int_0^1 \int_0^1 y(x^2 + 2y + 1) dx dy = \int_0^1 \int_0^1 x^2 y + 2y^2 + y dx dy$$

$$= \int_0^1 \left[\frac{1}{3} y + 2y^2 + y \right] dy = \frac{4}{3} \cdot \frac{1}{2} + \frac{2}{3} = \boxed{\frac{4}{3}}$$

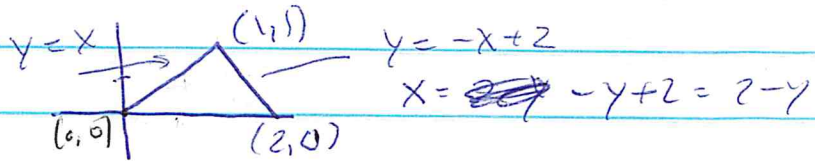
Thus,

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{5}{4}}{\frac{7}{3}} = \frac{15}{28} \approx 0.5357142$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{4}{3}}{\frac{7}{3}} = \frac{4}{7} = \frac{16}{28} \approx 0.571428571$$

Ex

Let R be the triangular region below. Let $p(x,y) = xy$ be a density function. Find the mass and center of mass.



$$\text{Mass} = \int_0^1 \int_y^{2-y} xy \, dx \, dy = \int_0^1 \frac{1}{2} x^2 y \Big|_y^{2-y} dy$$

$$\frac{1}{2} \int_0^1 (2-y)^2 y - y^3 \, dy = \frac{1}{2} \int_0^1 (4y - 4y^2 + y^3 - y^3) dy$$

$$= 2 \int_0^1 y - y^2 \, dy = 2 \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{2}{6} = \boxed{\frac{1}{3}}$$

$$M_x = \int_0^1 \int_y^{2-y} xy^2 \, dx \, dy = \dots = 2 \int_0^1 y^2 - y^3 \, dy = 2 \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{2}{12} = \boxed{\frac{1}{6}}$$

$$M_y = \int_0^1 \int_y^{2-y} x^2 y \, dx \, dy = \int_0^1 \frac{1}{3} x^3 y \Big|_y^{2-y} dy =$$

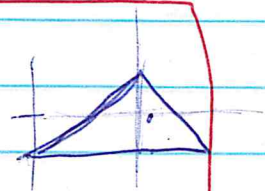
$$\frac{1}{3} \int_0^1 (2-y)^3 y - y^4 \, dy = \frac{1}{3} \int_0^1 [8 - 12y + 6y^2 - y^3] y - y^4 \, dy$$

$$= \frac{1}{3} \int_0^1 8y - 12y^2 + 6y^3 - 2y^4 \, dy = \frac{1}{3} \left(\frac{8}{2} - \frac{12}{3} + \frac{6}{4} - \frac{2}{5} \right)$$

$$= \frac{1}{3} \left(\frac{3}{2} - \frac{2}{5} \right) = \frac{1}{3} \left(\frac{15-4}{10} \right) = \boxed{\frac{11}{30}}$$

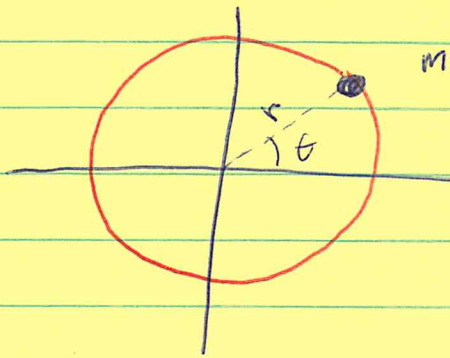
$$\bar{x} = \frac{M_y}{M} = \frac{\frac{11}{30}}{\frac{1}{3}} = \frac{11}{10} = 1.1$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2} = 0.5$$



Rotational Inertia

Idea



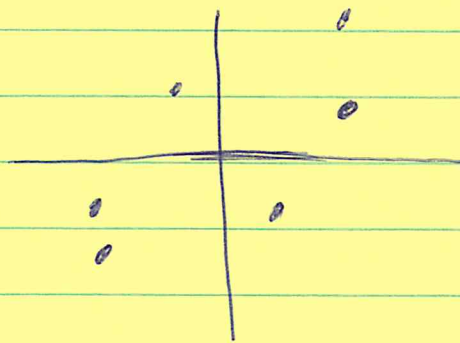
$$\text{Angular Velocity: } \omega = \frac{d\theta}{dt}$$

$$\text{Linear Velocity: } v = r\omega$$

$$\begin{aligned} \text{Kinetic Energy } E &= \frac{1}{2} m v^2 \\ &= \frac{1}{2} m r^2 \omega^2 \end{aligned}$$

↓
 I_0 (or I_z).

I_0 is called the rotational inertia.

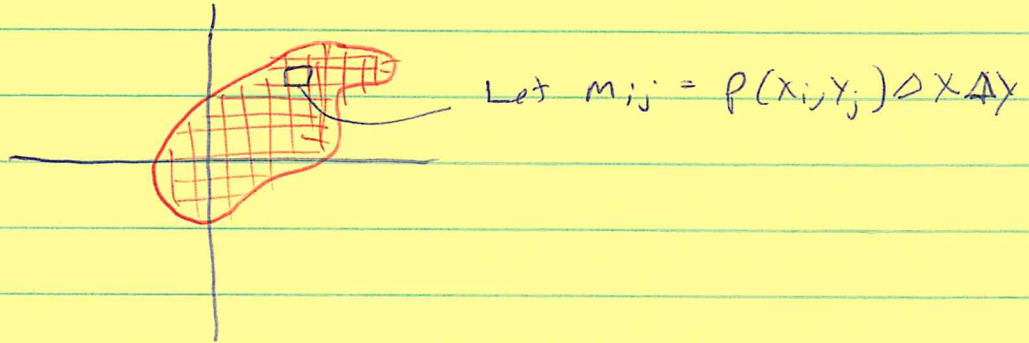


Suppose we have several objects rotating with the same ω about (o,o).

$$\begin{aligned} \text{Now } E &= \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i r_i^2 \omega_i^2 \\ &= \frac{1}{2} \left(\sum m_i r_i^2 \right) \omega^2 \\ &= \frac{1}{2} I_0 \omega^2 \end{aligned}$$

I_0 is the total rotational inertia

Now suppose we have an object in the xy -plane with density $\rho(x, y)$ and we want to find its rotational inertia wrt $(0, 0)$.



$$\text{Then } E \approx \frac{1}{2} \left(\sum \sum m_{ij} r_{ij}^2 \right) \omega^2$$

So in limit as $\Delta x, \Delta y \rightarrow 0$ we get

$$E = \frac{1}{2} \left(\iint r^2 \rho(x, y) dx dy \right) \omega^2.$$

We defined

$$I_0 = \iint_R r^2 \rho dA$$

could be

$$I_0 = \iint (x^2 + y^2) \rho(x, y) dx dy$$

or

$$I_0 = \iint r^3 \rho(r, \theta) dr d\theta$$

It is ~~convenient~~ convenient to ~~also~~ also define

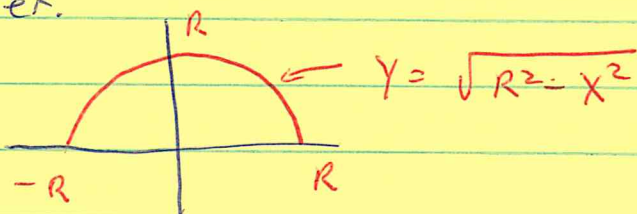
$$I_x = \iint x^2 \rho(x, y) dx dy$$

$$I_y = \iint y^2 \rho(x, y) dx dy$$

Then $I_0 = I_x + I_y$.

Ex

Find I_0 for the semi-circle with radius R and density proportional to the distance from the diameter, where I_0 means wrt to the circle's center.

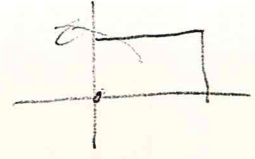


Sol

Let $\rho(x, y) = ky$, for some $k > 0$.

$$I_0 = \int_0^\pi \int_0^R k r \sin \theta r^2 dr d\theta$$

$$= k \frac{R^5}{5} \int_0^\pi \sin \theta d\theta = \frac{2kR^5}{5}$$



Ex Find I_0 for the rectangle $[0, 2] \times [0, 1]$ with $p(x, y) = 3x + 2y + 1$.

Sol:

$$\begin{aligned}
 I_x &= \int_0^1 \int_0^2 y^2 (3x + 2y + 1) dx dy \\
 &= \int_0^1 \int_0^2 3xy^2 + 2y^3 + y^2 dx dy \\
 &= \int_0^1 \left. \frac{3x^2y^2}{2} + (2y^3 + y^2)x \right|_0^2 dy \\
 &= \int_0^1 6y^2 + 4y^3 + 2y^2 dy \\
 &= \left. \frac{8y^3}{3} + y^4 \right|_0^1 = \frac{8}{3} + 1 = \frac{11}{3} = 3\frac{2}{3}
 \end{aligned}$$

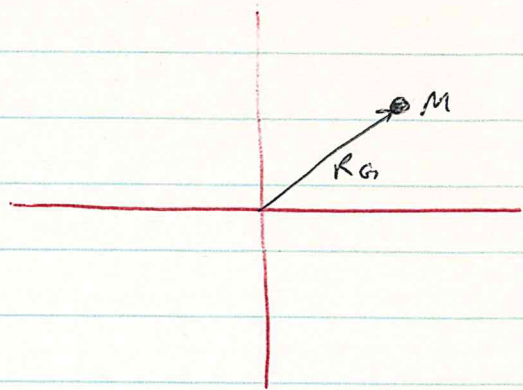
$$\begin{aligned}
 I_y &= \int_0^1 \int_0^2 x^2 (3x + 2y + 1) dx dy \\
 &= \int_0^1 \int_0^2 3x^3 + 2x^2y + x^2 dx dy = \\
 &= \int_0^1 \left(\frac{3 \cdot 16}{4} + \frac{2 \cdot 8y}{3} + \frac{8}{3} \right) dy = \int_0^1 \frac{16}{3}y + \frac{4y}{3} dy \\
 &= \frac{8}{3} + \frac{44}{3} = \frac{52}{3} = 17\frac{1}{3}
 \end{aligned}$$

$$I_0 = 3\frac{2}{3} + 17\frac{1}{3} = 21.$$

Note If the rect. is rotating about $(0,0)$ with angular vel. ω then $E = \frac{1}{2} I_0 \omega^2$.

Radius of Gyration.

$$\text{Let } R_G = \sqrt{\frac{I_0}{M}}$$



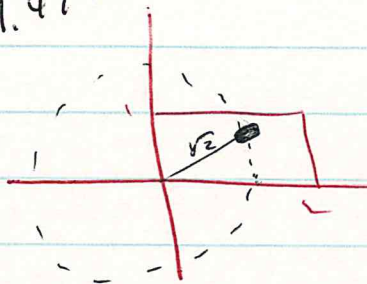
$$I = MR_G^2 = I_0$$

EX The radius of Gy of the last example is,

$$R_G = \sqrt{\frac{I_0}{M}}$$

$$\begin{aligned} M &= \int_0^1 \int_0^2 (3x+2y+1) dx dy = \\ &= \int_0^1 (6+4y+2) dy = 8+2 = 10 \end{aligned}$$

$$R_G = \sqrt{\frac{21}{10}} \approx 1.449137675\dots$$



~~$$I = \frac{1}{2} \cdot (10) \cdot (2)^2 = 100$$~~

Ex A disk with radius 1 meter and mass 3 kg is moving 10 m/s in a straight line and spinning 5 revolutions per second. It has uniform density. Find the total kinetic energy.

Sol $E = \frac{1}{2}mv^2 + \frac{1}{2}I_0\omega^2$. We know m and v , and $\omega = 5 \frac{\text{rev}}{\text{sec}} = 5 \frac{\text{rev}}{\text{sec}} \cdot 2\pi \frac{\text{radians}}{\text{rev}} = 10\pi \frac{\text{radians}}{\text{sec}}$. We just need to find I_0 .

$$I_0 = \int_0^{2\pi} \int_0^1 r^2 \rho r dr d\theta. \quad \rho = \frac{\text{mass}}{\text{area}} = \frac{3 \text{ kg}}{\pi \text{ m}^2}$$

$$I_0 = \frac{3}{\pi} \int_0^{2\pi} \int_0^1 r^3 dr d\theta = \frac{3}{\pi} \cdot \frac{1}{4} \cdot 2\pi = \frac{3}{2}.$$

$$\begin{aligned} \text{Thus, } E &= \frac{1}{2} \cdot 3(10)^2 + \frac{1}{2} \left(\frac{3}{2}\right) (10\pi)^2 = 150 + 75\pi^2 \\ &\approx 846.22 \text{ J.} \\ &\quad \uparrow \\ &\quad \text{Joules} \\ &= \text{kg} \frac{\text{m}^2}{\text{s}^2} \end{aligned}$$

Additional Examples if Time Permits

Ex Let R be the square $[0, 1] \times [0, 1]$. Let $\rho(x, y) = xy + 1$. Find the mass, the center of mass, the rotational inertia wrt to $(0, 0)$ and the radius of gyration.



Sol $M = \int_0^1 \int_0^1 xy + 1 \, dx \, dy = \int_0^1 \frac{1}{2}y + 1 \, dy = \frac{1}{4} + 1 = \frac{5}{4}$.

$$M_x = \int_0^1 \int_0^1 y(xy + 1) \, dx \, dy = \int_0^1 \int_0^1 xy^2 + y \, dx \, dy = \int_0^1 \frac{1}{2}y^2 + y \, dy$$
$$= \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$$

$M_y = M_x$ by symmetry. Look for short cuts like this!

Thus $\bar{x} = \bar{y} = \frac{2/3}{5/4} = \frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$. The C.M. = $(\frac{8}{15}, \frac{8}{15})$.
↑ center of mass.

$$I_x = \int_0^1 \int_0^1 y^2(xy + 1) \, dx \, dy = \int_0^1 \int_0^1 xy^3 + y^2 \, dx \, dy$$
$$= \int_0^1 \frac{1}{2}y^3 + y^2 \, dy = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} = \frac{11}{24}$$

$I_y = I_x$ by symmetry. Thus, $I_0 = \frac{11}{24} + \frac{11}{24} = \frac{22}{24} = \frac{11}{12}$.

$$R_G = \sqrt{\frac{I_0}{M}} = \sqrt{\frac{11/12}{5/4}} = \sqrt{\frac{11}{15}} \approx 0.8563$$

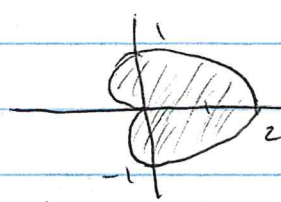
Done!

Ex

Let R be the region determined by $0 \leq r \leq 1 + \cos \theta$.
The density is proportional to the distance from
the x -axis. Find M , \bar{x} , \bar{y} , I_0 , and R_G .

Sol

Graph the region.



It is a cardioid -
a heart shape.

Find the density function. $\rho(x, y) = k|y| = kr|\sin \theta|$.
↑
for some k

Notice both the region and the density are
symmetric across the x -axis.

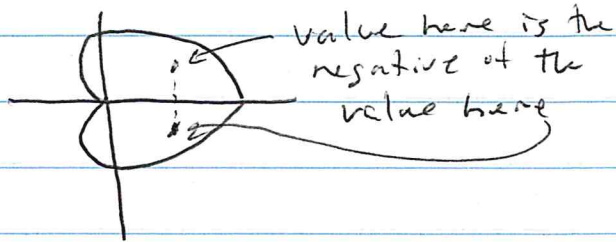
$$\begin{aligned} M &= \int_0^{2\pi} \int_0^{1+\cos\theta} kr|\sin\theta| r dr d\theta = k \int_0^{2\pi} \int_0^{1+\cos\theta} r^2 |\sin\theta| dr d\theta \\ &= 2k \int_0^{\pi} \int_0^{1+\cos(\theta)} r^2 \sin\theta dr d\theta = 2k \int_0^{\pi} \frac{r^3}{3} \sin\theta \Big|_0^{1+\cos\theta} d\theta \\ &= \frac{2k}{3} \int_0^{\pi} (1+\cos\theta)^3 \sin\theta d\theta. \end{aligned}$$

Let $u = 1 + \cos \theta$. Then $du = -\sin \theta d\theta$
and u goes from 2 to 0. Thus,

$$M = \frac{-2k}{3} \int_2^0 u^3 du = \frac{2k}{3} \int_0^2 u^3 du = \frac{2k}{3} \frac{16}{4} = \boxed{\frac{8}{3}k}$$

$$M_x = \int_0^{2\pi} \int_0^{1+\cos\theta} \underbrace{(r \sin\theta)}_y k r |\sin\theta| r dr d\theta = \boxed{0}$$

Why? This integrand is odd w.r.t. y .



$$M_y = \int_0^{2\pi} \int_0^{1+\cos\theta} \underbrace{(r \cos\theta)}_x k r |\sin\theta| r dr d\theta$$

$$= 2k \int_0^{\pi} \int_0^{1+\cos\theta} r^3 \cos\theta \sin\theta dr d\theta$$

$$= \frac{2k}{4} \int_0^{\pi} (1+\cos\theta)^4 \cos\theta \sin\theta d\theta$$

Let $u = 1 + \cos\theta$. Then $du = -\sin\theta d\theta$ and u goes from 2 to 0. Also notice $\cos\theta = u - 1$. Thus

$$M_y = -\frac{k}{2} \int_2^0 u^4 (u-1) du = \frac{k}{2} \int_0^2 u^5 - u du$$

$$= \frac{k}{2} \left(\frac{2^6}{6} - \frac{2^2}{2} \right) = k \left(\frac{2^4}{3} - 1 \right) = k \left(5\frac{1}{3} - 1 \right)$$

$$= \boxed{\frac{13}{3} k}$$

~~C.A. is 0,~~

$$\bar{x} = \frac{0}{M} = 0. \quad \bar{y} = \frac{M_x}{M} = \frac{\frac{13}{3}k}{\frac{8}{3}k} = \frac{13}{8}$$

$$\text{C.M.} = \left(0, \frac{13}{8}\right).$$

→ We know by symmetry.

$$I_0 = k \int_0^{2\pi} \int_0^{1+\cos\theta} r^2 r \sin\theta \, r \, dr \, d\theta =$$

$$2k \int_0^{\pi} \int_0^{1+\cos\theta} r^4 \sin\theta \, dr \, d\theta =$$

$$\frac{2k}{5} \int_0^{\pi} (1+\cos\theta)^5 \sin\theta \, d\theta.$$

Let $u = 1 + \cos\theta$. Then $du = -\sin\theta \, d\theta$, $u = 2$ to 0 .

$$I_0 = -\frac{2k}{5} \int_2^0 u^5 \, du = \frac{2}{5}k \int_0^2 u^5 \, du = \frac{2}{k} \frac{u^6}{6} \Big|_0^2 = \boxed{\frac{64}{15}k}.$$

$$R_G = \sqrt{\frac{I_0}{M}} = \sqrt{\frac{\frac{64}{15}k}{\frac{8}{3}k}} = \sqrt{\frac{16}{5}} = \frac{4}{\sqrt{5}} \approx \boxed{1.78885}$$

Done!