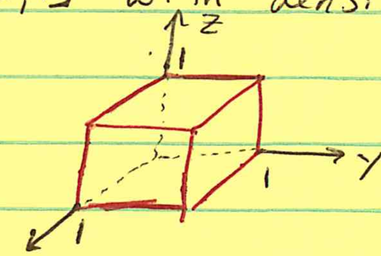
12.5 Triple Integrals

Ex Find the mass of the cube $[0,1]^3$ with density function $\rho(x,y,z) = x^2yz$.

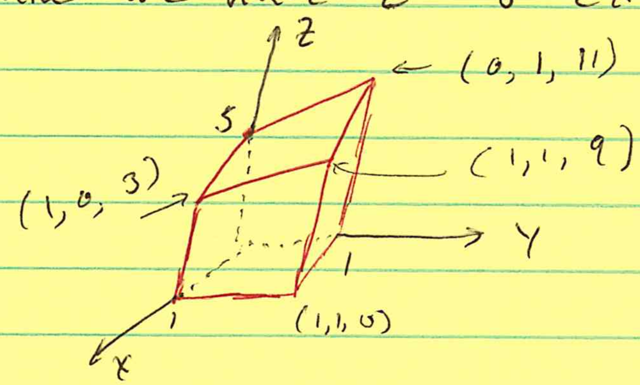


Sol $\text{Mass} = \int_0^1 \int_0^1 \int_0^1 x^2yz \, dx\,dy\,dz$

$$= \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2} = \boxed{\frac{1}{12}}.$$

Ex Find the mass of the region under the plane given by $2x - 6y + z = 5$ and over the square $[0,1]\times[0,1]$ in the xy-plane with density given by $\rho(x,y,z) = e^z$.

Sol For the plane we have $z = 5 - 2x + 6y$.



$$\text{Mass} = \int_0^1 \int_0^1 \int_0^{5-2x+6y} e^z \, dz\,dx\,dy$$

$$= \int_0^1 \int_0^1 e^{5-2x+6y} - \underbrace{e^0}_{1} \, dx\,dy$$

$$= \int_0^1 \int_0^1 e^{5+6y} e^{-2x} - 1 \, dx \, dy$$

$$= \int_0^1 \left(-\frac{1}{2} e^{5+6y} e^{-2x} - x \right)\Big|_0^1 \, dy$$

$$= \int_0^1 \left(-\frac{1}{2} e^{3+6y} - 1 \right) - \left(-\frac{1}{2} e^{5+6y} - 0 \right) \, dy$$

$$= \int_0^1 \frac{1}{2} (e^5 - e^3) e^{6y} - 1 \, dy$$

$$= \frac{1}{12} (e^5 - e^3) e^{6y} - y \Big|_0^1$$

$$= \frac{1}{12} (e^5 - e^3) e^6 - 1 - \frac{1}{12} (e^5 - e^3)$$

$$= \boxed{\frac{(e^5 - e^3)(e^6 - 1)}{12} - 1 \approx 4302.560847}$$

The next page shows how to use 3 diffrent computer programs to do this integral.

Maple

```
> int(int(int(exp(z),z=0..5-2*x+6*y),x=0..1),y=0..1);
```

$$-\frac{1}{12}e^5+\frac{1}{12}e^3+\frac{1}{12}e^{11}-1-\frac{1}{12}e^9 \quad (1)$$

```
> evalf(%);
```

$$4302.560847 \quad (2)$$

```
>
```


wxMaxima document

(%i2) integrate(integrate(integrate(exp(z),z,0,5−2·x+6·y),x,0,1),y,0,1);

(%o2) $$\frac{\%e^{11}-\%e^9-12}{12}-\frac{\%e^5-\%e^3}{12}$$

(%i3) float(%);

(%o3) 4302.560847120253

WolframAlpha computational intelligence

triple integral calculator

Extended Keyboard   Upload   Examples   Random

Computational Inputs:

- » function to integrate: exp(z)
- » innermost variable: z
- » innermost lower limit: 0
- » innermost upper limit: 5-2*x+6*y
- » middle variable: x
- » middle lower limit: 0
- » middle upper limit: 1
- » outermost variable: y
- » outermost lower limit: 0
- » outermost upper limit: 1

Compute

Definite integral:

More digits

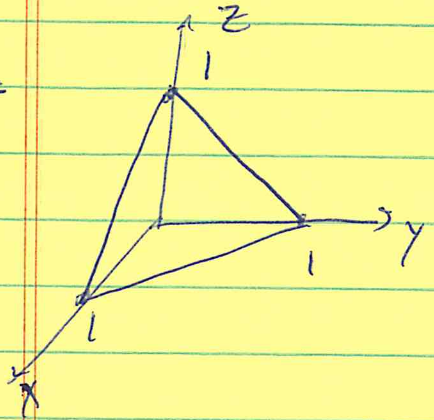
$$\int_0^1\int_0^1\int_0^{5-2x+6y}\exp(z)\,dz\,dx\,dy=\frac{1}{12}\left(-12+e^3-e^5-e^9+e^{11}\right)\approx 4302.56$$

Open code

Ex Let $\rho(x,y,z) = x$ be the density of the region in the first octant under the plane given by

$$x + y + z = 1.$$

Sol.



You can set up the integral six different ways.

$dx\,dy\,dz$
$dx\,dz\,dy$
$dy\,dx\,dz$
$dy\,dz\,dx$
$dz\,dx\,dy$
$dz\,dy\,dx$.

I'll do $dx\,dy\,dz$ here and two others on the next page. You should do the other three

$dx\,dy\,dz$

$$M = \int_0^1 \int_0^{1-z} \int_0^{1-y-z} x \, dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-z} \frac{1}{2}(1-y-z)^2 \, dy\,dz.$$

Let $u = 1-y-z$. Then $du = -dy$. And u goes from $1-z$ to 0. Thus,

$$M = -\frac{1}{2}\int_0^1 \int_{1-z}^{0} u^2 \, du\,dz = -\frac{1}{2}\int_0^1 \frac{u^3}{3}\Bigg|_{1-z}^{0} dz$$

$$= \frac{1}{6}\int_0^1 (1-z)^3 \, dz$$

Let $u = 1-z$. Then $du = -dz$. And u goes from 1 to 0

$$M = -\frac{1}{6}\int_1^0 u^3\,du = \frac{1}{6}\int_0^1 u^3\,du = \frac{1}{6}\cdot\frac{1}{4} = \boxed{\frac{1}{24}}.$$

dy dz dx

$$M = \int_0^1 \int_0^{1-x} \int_0^{1-x-z} x \, dy \, dz \, dx$$

$$= \int_0^1 \int_0^{1-x} xy \Big|_{y=0}^{y=1-x-z} dz \, dx$$

$$= \int_0^1 \int_0^{1-x} x - x^2 - xz \, dz \, dx$$

$$= \int_0^1 x(1-x) - x^2(1-x) - \tfrac{1}{2}x(1-x)^2 \, dx$$

$$= \int_0^1 \tfrac{1}{2}x^3 - x^2 + \tfrac{1}{2}x \, dx$$

$$= \frac{1}{8} - \frac{1}{3} + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{8} - \frac{1}{3} = \boxed{\frac{1}{24}}$$

dz dx dy

$$M = \int_0^1 \int_0^{1-y} \int_0^{1-x-y} x \, dz \, dx \, dy$$

$$= \int_0^1 \int_0^{1-y} x - x^2 + xy \, dx$$

$$= \int_0^1 \tfrac{1}{2}(1-y)^2 - \tfrac{1}{3}(1-y)^3 - \tfrac{1}{2}(1-y)^2 y \, dy$$

$$= \int_0^1 \frac{1-2y+y^2}{2} - \frac{1-3y+3y^2-y^3}{3} - \frac{y-2y^2+y^3}{2} \, dy$$

$$= \int_0^1 -\tfrac{1}{6}y^3 + \tfrac{1}{2}y^2 - \tfrac{1}{2}y \, dy$$

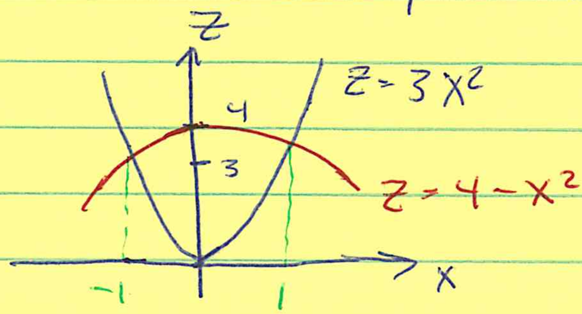
$$= -\frac{1}{24} + \frac{1}{3} - \frac{1}{4} = \frac{-1}{24} + \frac{1}{12} = \boxed{\frac{1}{24}}$$

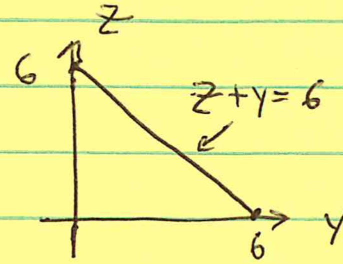
Ex (From Swokowski, page 678).

Find the volume in $\mathbb{R}^3$ of the region bounded by

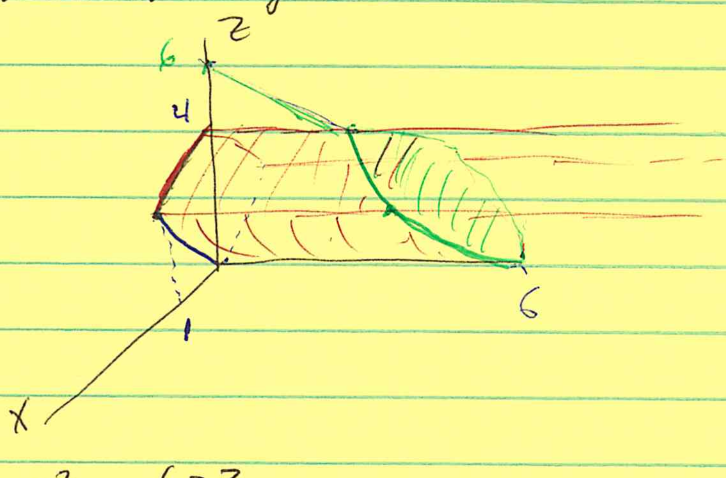
$$z = 3x^2$$
$$z = 4 - x^2$$
$$y = 0 \longleftarrow \text{xz-plane}$$
$$z + y = 6.$$

Sol Draw lots of pictures.





Slowly put them together



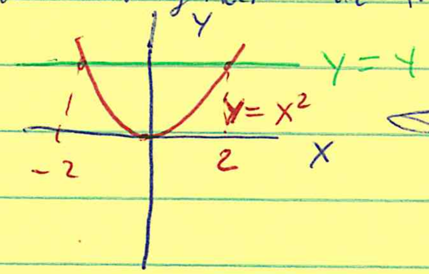
$$\text{Vol} = \int_{-1}^{1} \int_{3x^2}^{4-x^2} \int_{0}^{6-z} 1 \, dy\, dz\, dx = \cdots = \boxed{\frac{304}{15}}$$

Ex Find the volume bounded by

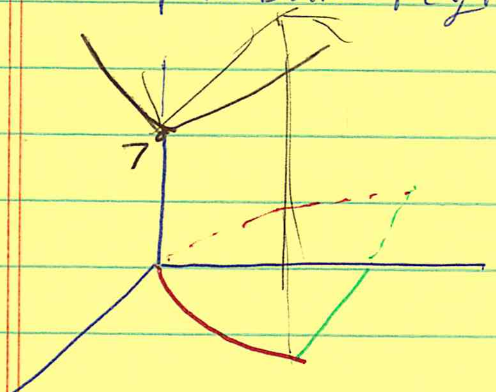
$$z = 3x^2 + y^2 + 7$$
$$y = x^2$$
$$z = 0 \quad \leftarrow \text{xy-plane}$$
$$y = 4 \quad \leftarrow \text{a plane parallel to xz-pla}$$

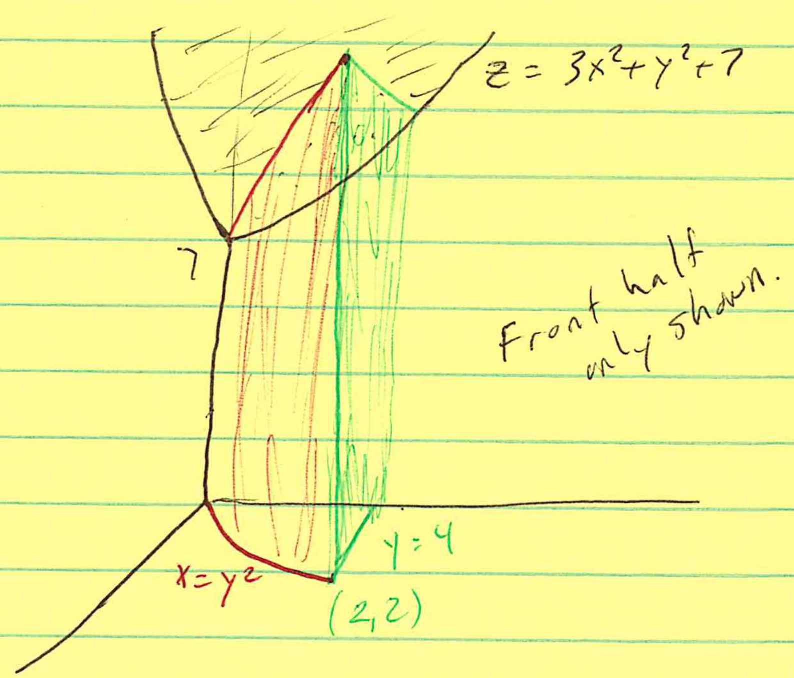
Sol We are going from the xy-plane ($z=0$) up to the paraboloid $z = 3x^2 + y^2 + 7$.

The equation $y = x^2$, $y = 4$ determine a bounded region in the xy-plane.



So, the solid region is below the paraboloid over the parabolic region in the xy-plane.





$$\text{Vol.} = \int_{-2}^{2} \int_{x^2}^{4} \int_{0}^{3x^2+y^2+7} 1 \; dz \, dy \, dx$$

$$= 2 \int_{0}^{2} \int_{x^2}^{4} 3x^2 + y^2 + 7 \; dy \, dx$$

using symmetry

$$= 2 \int_{0}^{2} 3x^2 y + \frac{y^3}{3} + 7y \Big|_{x^2}^{4} \, dx$$

$$= 2 \int_{0}^{2} \left(12x^2 + \frac{64}{3} + 28\right) - \left(3x^4 + \frac{x^6}{3} + 7x^2\right) \, dx$$

$$= 2 \int_{0}^{2} -\frac{x^6}{3} - 3x^4 + 5x^2 + \frac{148}{3} \, dx$$

$$= 2 \left[-\frac{x^7}{21} - \frac{3x^5}{5} + \frac{5x^3}{3} + \frac{148}{3} x \right]_{0}^{2}$$

$$= 2 \left[-\frac{128}{21} - \frac{96}{5} + \frac{40}{3} + \frac{296}{3} \right] = \frac{18{,}208}{105}$$

$$\approx 173.409523$$

Ex Integrate $f(x,y,z) = xy + yz + xz$ over the solid ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} \le 1.$$

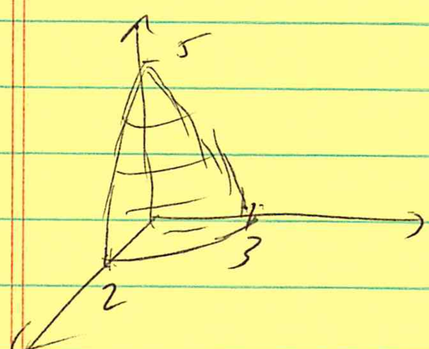
Sol Use symetry. The answer is 0. Think about this. Compare to

$$\int\int xy \, dx dy$$

over a disk or ellipse with center $(0,0)$.

Ex Find the volume of the solid ellipsoid above.

Sol



$$\text{Vol.} = 8 \int_0^5 \int_0^{3\sqrt{1-z^2/25}} \int_0^{2\sqrt{1-\frac{y^2}{9}-\frac{z^2}{25}}} 1 \, dx dy dz.$$

$$= 16 \int_0^5 \int_0^{3\sqrt{1-\frac{z^2}{25}}} \sqrt{1-\frac{y^2}{9}-\frac{z^2}{25}} \, dy \, dz$$

Use a trig sub. Treat z as a constant for now.

Let $y = 3\sqrt{1-\frac{z^2}{25}}\,\sin\theta, \quad 0 \le \theta \le \frac{\pi}{2}$.

Then $\sqrt{1-\frac{z^2}{25}-\frac{y^2}{9}} = \sqrt{\left(1-\frac{z^2}{25}\right)-\left(1-\frac{z^2}{25}\right)\sin^2\theta}$

$= \sqrt{1-\frac{z^2}{25}}\,\cos\theta.$

Also, $dy = 3\sqrt{1-\frac{z^2}{25}}\,\cos\theta\, d\theta.$

Thus,

$$\text{Vol} = 16\int_0^5\int_0^{\pi/2} 3\left(1-\frac{z^2}{25}\right)\cos^2\theta\, d\theta\, dz$$

$\boxed{\text{You check that } \int_0^{\pi/2}\cos^2\theta\, d\theta = \pi/4.}$

$$= 12\pi\int_0^5 1-\frac{z^2}{25}\, dz$$

$$= 12\pi\left(z-\frac{z^3}{75}\right)\Big|_0^5 = 12\pi\left(5-\underbrace{\frac{5}{3}}_{\frac{15}{3}}\right)$$

$$= 40\pi$$

Note The vol. inside the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

is $\quad V = \frac{4}{3}abc\pi.$

Applications of Triple Integrals

$$M = \text{Mass} = \iiint \rho(x, y, z)\, dV.$$

First Moments wrt to coordinate planes:

$$M_{yz} = \iiint x\rho \, dV$$

$$M_{xz} = \iiint x\rho \, dV$$

$$M_{xy} = \iiint z\rho \, dV$$

Center of mass: $\bar{x} = \dfrac{M_{yz}}{M}$, $\bar{y} = \dfrac{M_{xz}}{M}$, $\bar{z} = \dfrac{M_{xy}}{M}$.

Second Moments wrt coordinate axes:

$$I_x = \iiint (y^2+z^2)\rho \, dV$$

$$I_y = \iiint (x^2+z^2)\rho \, dV$$

$$I_z = \iiint (x^2+y^2)\rho \, dV$$

Radii of Gyration

$$\bar{\bar{x}} = \sqrt{\frac{I_x}{M}} \qquad \bar{\bar{y}} = \sqrt{\frac{I_y}{M}} \qquad \bar{\bar{z}} = \sqrt{\frac{I_z}{M}}$$

Ex Find the center of mass of the cube $[0,1]^3$ with density $\rho(x,y,z) = xyz$.

Sol

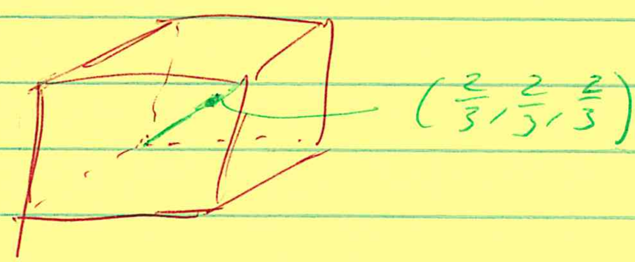
$$M = \int_0^1\int_0^1\int_0^1 xyz\,dx\,dy\,dz = \frac{1}{8}$$

$$M_{yz} = \iiint x(xyz)\,dx\,dy\,dz = \frac{1}{12}$$

$$M_{xz} = \iiint y(xyz)\,dx\,dy\,dz = \text{same} = \frac{1}{12}$$

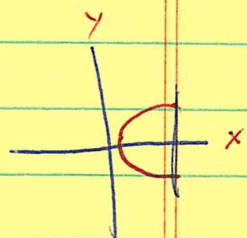
$$M_{xy} = \iiint z(xyz)\,dx\,dy\,dz = \text{same} = \frac{1}{12}$$

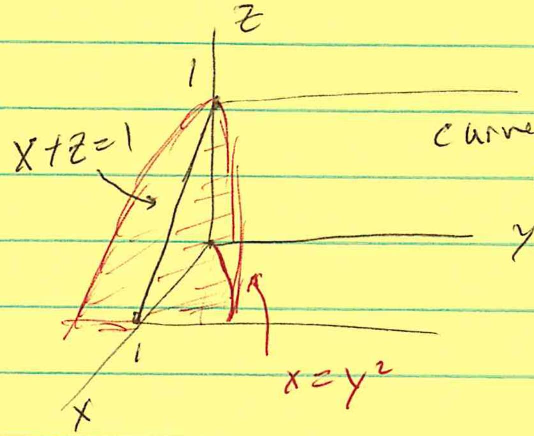
Thus $\bar{x} = \bar{y} = \bar{z} = \dfrac{\frac{1}{12}}{\frac{1}{8}} = \dfrac{8}{12} = \dfrac{2}{3}$



Ex Find the volume and centroid of the region R bounded by $x=y^2$, $z=0$, $x+z=1$.

Sol $z=0$ is the xy-plane.





curved wedge.
Like a door stop.

$$V = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x} 1 \, dz\,dx\,dy = \cdots = \frac{8}{15}$$

Centroid Use $\rho = 1$.

$$\bar{x} = \frac{1}{V}\iiint x \, dV = \frac{15}{8}\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x} x \, dz\,dx\,dy$$

$$= \cdots = \frac{3}{7}$$

Clearly $\bar{y} = 0$.

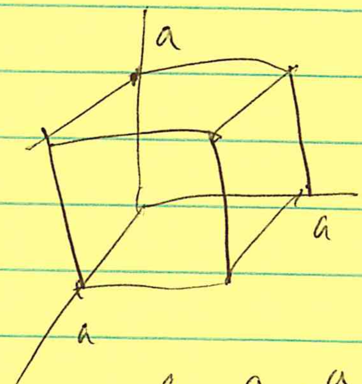
$$\bar{z} = \frac{1}{V}\iiint z \, dV = \frac{15}{8}\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x} z \, dz\,dx\,dy$$

$$= \cdots = \frac{2}{7}.$$

$$\boxed{\text{Centroid is } \left(\frac{3}{7}, 0, \frac{2}{7}\right).}$$

Ex ⓐ Find the moment of interia and radius of gyration of a cube of edge length a with uniform density k with respect to an edge.

Sol



I'll use the z-axis as the edge.

Mass = density × vol. = ka^3.

$$I_z = \int_0^a \int_0^a \int_0^a (x^2+y^2)\,k \; dx\,dy\,dz$$

$$= k \int_0^a \int_0^a \left(\frac{a^3}{3} + y^2 a\right) dy\, dz$$

$$= k \int_0^a \frac{a^3}{3}a + \frac{a^3}{3}a \; dz$$

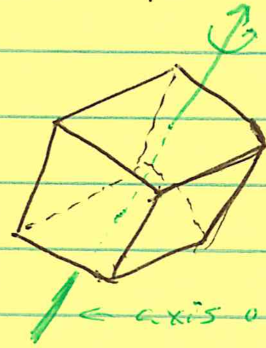
$$= \frac{2ka^4}{3} \int_0^a dz = \frac{2ka^5}{3}$$

$$R_g \text{ or } \bar{\bar{z}} = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{\frac{2ka^5}{3}}{ka^3}} = a\sqrt{\frac{2}{3}}$$

ⓑ If a = 1 meter, k = 3 kg/m³ and ω = 3 rev/sec what is the kinetic energy?

Sol $K.E. = \frac{1}{2} I_z \omega^2 = \frac{1}{2}\left(\frac{2\cdot 3\cdot 1^5}{3}\right)(3\cdot 2\pi)^2 = 36\pi^2 \approx 355.3$ Joules

Ex (a) Find the moment of intertia and radius of gyration of a cube of edge length a with uniform density k with resp. to a line down the middle as shown



Sol Mass $= k a^3$

$$I_z = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+y^2)k \, dx\,dy\,dz = \cdots\cdots = \frac{k a^5}{6}$$

[z could go from 0 to a and give the same result.]

$$R_G = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{k a^5/6}{k a^3}} = \frac{a}{\sqrt{6}}$$

(b) If $a = 1$ meter, $k = 3\ \text{kg}/m^3$ and $\omega = 3$ rev/sec what is the kinetic energy?

Sol $K.E. = \frac{1}{2} I_z \omega^2 = \frac{1}{2}\left(\frac{3 \cdot 1^5}{6}\right)(3 \cdot 2\pi) = \underline{9\pi^2 \approx 88.8 \text{ Joules}}$