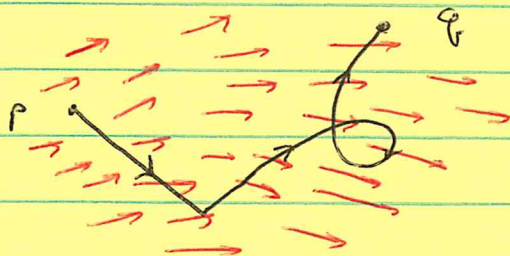


13.3

Path Independence and the Fundamental Thm of Line Integrals

Thm (FTLI) Let C be a piecewise smooth path with starting point $p = (x_1, y_1, z_1)$ and ending point $q = (x_2, y_2, z_2)$. Let F be a vector field with continuous gradient in an open region containing C .



If F is conservative then

$$\int_C F \cdot T \, ds = \int_C \nabla f \cdot T \, ds = f(q) - f(p)$$

where $f(x, y, z)$ is a potential function for F , $F = \nabla f$.

This means that it does not matter what path we use to get from p to q . In particular, for any closed loop C

$$\oint_C F \cdot T \, ds = 0$$

\oint_C is a loop.

Pf: Assume for now that C is smooth and that $r(t)$ is a parameterization for C .

$$\int_C F \cdot T ds = \int_a^b \nabla f \cdot \frac{dr}{dt} dt = \int_a^b f_x x' + f_y y' + f_z z' dt$$

$$\begin{aligned} \text{But } \frac{d f(x(t), y(t), z(t))}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \\ &= f_x x' + f_y y' + f_z z'. \end{aligned}$$

Thus,

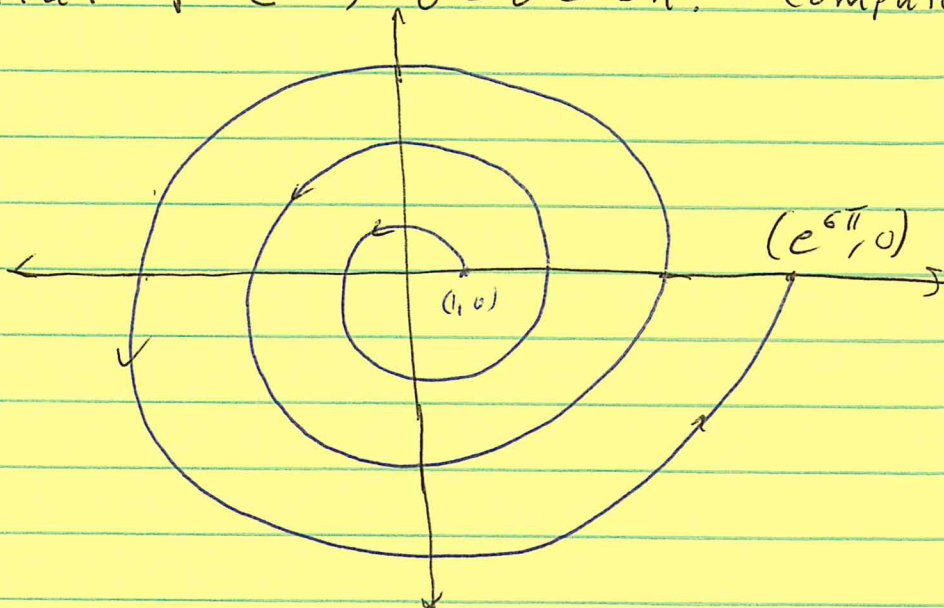
$$\begin{aligned} \int_C F \cdot T ds &= \int_a^b \frac{d}{dt} f(x(t), y(t), z(t)) dt \\ &= f(x(t), y(t), z(t)) \Big|_a^b = f(x(b), y(b), z(b)) - \\ & \qquad \qquad \qquad f(x(a), y(a), z(a)) \\ &= f(q) - f(p). \end{aligned}$$



Can generalize to piecewise smooth curves one piece at a time.

Ex

Let $F(x,y) = \langle 3y, 3x+y \rangle$. Let C be the spiral $r = e^\theta$, $0 \leq \theta \leq 6\pi$. Compute $\int_C F \cdot T ds$.



We will do this problem ~~4~~ ways.

Method I (Old Way) We parameterize the curve. I'll use θ as the parameter.

$$r(\theta) = \langle r \cos \theta, r \sin \theta \rangle = \langle e^\theta \cos \theta, e^\theta \sin \theta \rangle.$$

$$\text{Now } \int_C F \cdot T ds = \int_0^{6\pi} F \cdot \frac{dr}{d\theta} d\theta.$$

$$r'(\theta) = \langle e^\theta \cos \theta - e^\theta \sin \theta, e^\theta \sin \theta + e^\theta \cos \theta \rangle$$

$$F(\theta) = \langle 3e^\theta \sin \theta, 3e^\theta \cos \theta + e^\theta \sin \theta \rangle$$

$$F \cdot r' = e^{2\theta} (3sc - 3s^2 + 3sc + 3c^2 + s^2 + sc)$$

Thus,

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_0^{6\pi} e^{2\theta} (7sc - 2s^2 + 3c^2) \, d\theta$$

Computer integration gives 0.

Thus, $\int_C \mathbf{F} \cdot \mathbf{T} \, ds = 0.$

Method II

Check to see if the vector field is conservative.

$$\frac{\partial}{\partial y}(3y) = 3 \quad \frac{\partial}{\partial x}(3x+y) = 3. \quad \text{Hence conservative.}$$

Now find $f(x,y)$ with $\mathbf{F} = \nabla f$.

$$f = \int 3y \, dx = 3xy + C_1(y)$$

$$f = \int 3x+y \, dy = 3xy + \frac{1}{2}y^2 + C_2(x).$$

Let $f(x,y) = 3xy + \frac{1}{2}y^2$. You can check this works.

By the FTLI $\int_C \mathbf{F} \cdot \mathbf{T} \, ds = f(e^{6\pi}, 0) - f(1, 0) = 0 - 0 = 0.$

Method III

Since F is conservative it is path independent. So, we can use an easier path, and not have to find f .

$$\text{Let } r(t) = \langle (e^{6\pi} - 1)t + 1, 0 \rangle. \quad 0 \leq t \leq 1$$

$$\text{Now } r'(t) = \langle e^{6\pi} - 1, 0 \rangle$$

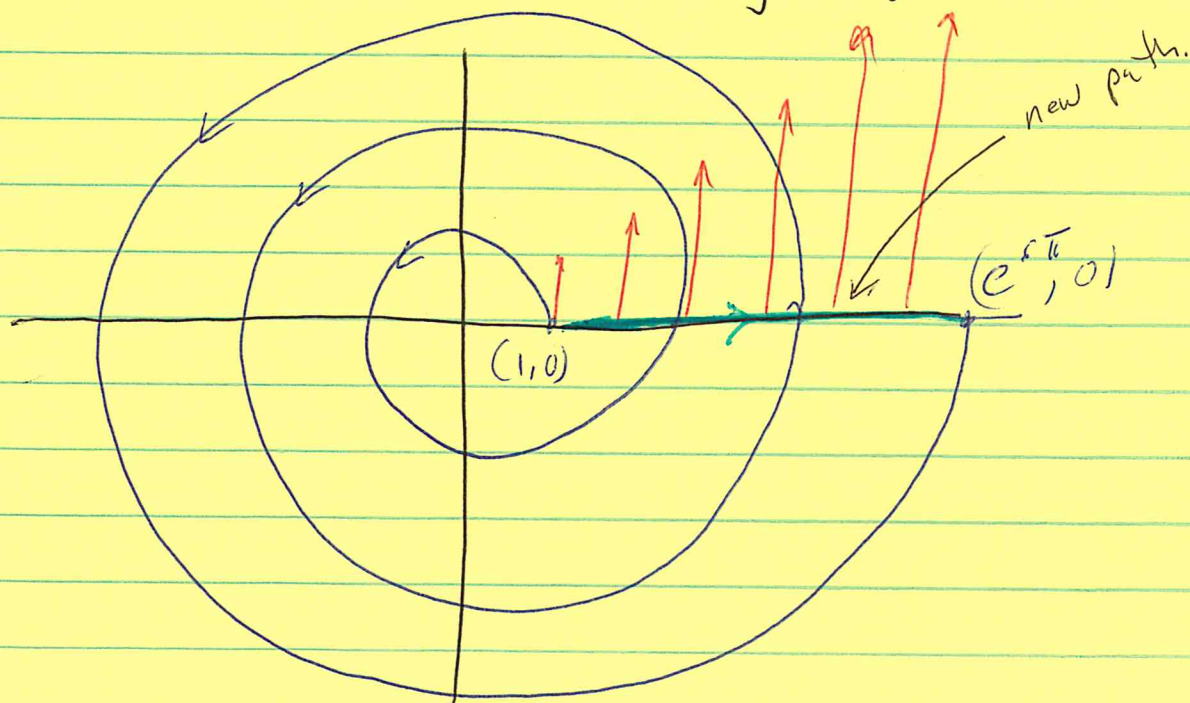
$$\begin{aligned} \text{and } F(t) &= \langle 3 \cdot 0, 3(e^{6\pi} - 1)t + 3 + 0 \rangle \\ &= \langle 0, 3(e^{6\pi} - 1)t + 3 \rangle. \end{aligned}$$

$$F \cdot r' = 0 \cdot (e^{6\pi} - 1) + 3 \cdot 0 = 0.$$

$$\text{Thus } \int F \cdot T ds = \int_0^1 F \cdot r' dt = \int_0^1 0 dt = 0.$$

Method IV

Look at the picture. The path from III is \perp to the vector field. Thus $\int F \cdot T ds = 0$.



Ex Let $F = \langle 2x+z, 2y, x \rangle$.

Let C be the path $r(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 1$.

Find $\int_C F \cdot T ds$.

Sol I

$$\begin{aligned}\int_C F \cdot T ds &= \int_0^1 F \cdot \frac{dr}{dt} dt = \int_0^1 \langle 2t+t^3, 2t^2, t \rangle \cdot \langle 1, 2t, 3t^2 \rangle dt \\ &= \int_0^1 2t + t^3 + 4t^3 + 3t^3 dt \\ &= \int_0^1 2t + 8t^3 dt = 2 \cdot \frac{1}{2} + 8 \cdot \frac{1}{4} = 1 + 2 = \underline{3}.\end{aligned}$$

Sol II

I claim F is conservative. We do not have a test for this (yet) but if I can find a potential function, then F is conservative.

$$f = \int 2x+z dx = x^2 + xz + C_1(y, z)$$

$$f = \int 2y dy = y^2 + C_2(x, z)$$

$$f = \int x dz = xz + C_3(x, y)$$

Let $f(x, y, z) = x^2 + y^2 + xz$. Check that $\nabla f = F$.

By the FTLI $\int_C F \cdot T ds = f(1, 1, 1) - f(0, 0, 0) = 3 - 0 = \underline{3}$.

Ex Let C be the unit circle, $x^2 + y^2 = 1$, counter clockwise starting at $(0, 1)$. Compute

$$\oint_C (2xy) dx + (x^2 + 2x) dy.$$

Sol Step 1 Check if conservative. If the vector field $\langle 2xy, x^2 + 2x \rangle$ is cons. the answer is 0.

$$\partial_y(2xy) = 2x, \quad \partial_x(x^2 + 2x) = 2x + 2.$$

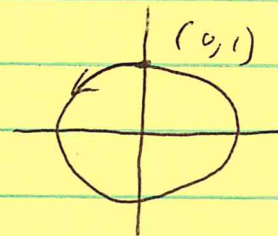
It is not cons. but it almost is! Watch.

$$\oint_C (2xy) dx + (x^2 + 2x) dy = \underbrace{\oint_C (2xy) dx + (x^2) dy}_{=0} + \oint_C 2x dy$$

So, we only need to compute $\oint_C 2x dy$.

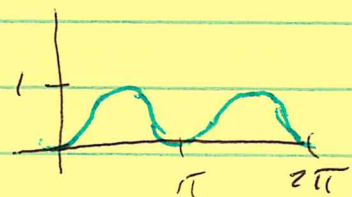
Step 2 Parametrize C .

Use $r(t) = \langle -\sin t, \cos t \rangle$



Step 3 Do the integration.

$$\begin{aligned} \oint_C 2x dy &= \int_0^{2\pi} 2x(t) \frac{dy}{dt} dt = \int_0^{2\pi} (-2\sin t)(-\sin t) dt \\ &= 2 \int_0^{2\pi} \sin^2 t dt = 2\pi \end{aligned}$$



Thm The 3 following conditions are equivalent.

1. F is conservative

2. $\int_C F \cdot T ds$ or $\int_C F \cdot dr$ or $\int_a^b F \cdot \frac{dr}{dt} dt$

only dependent on the end points. Specifically

$$\int_C F \cdot T ds = f(b) - f(a) \quad \text{where } F = \nabla f.$$

3. $\oint_C F \cdot T ds = 0$ for all closed curves.