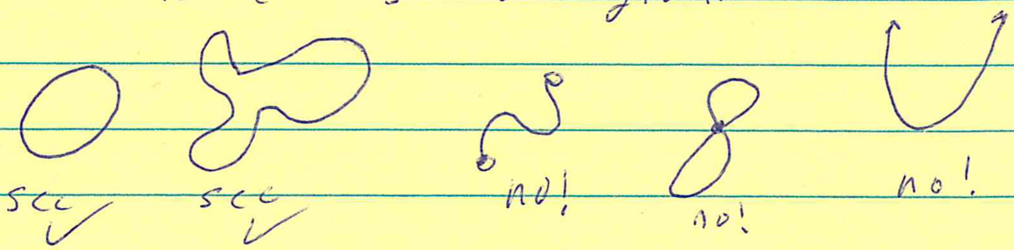
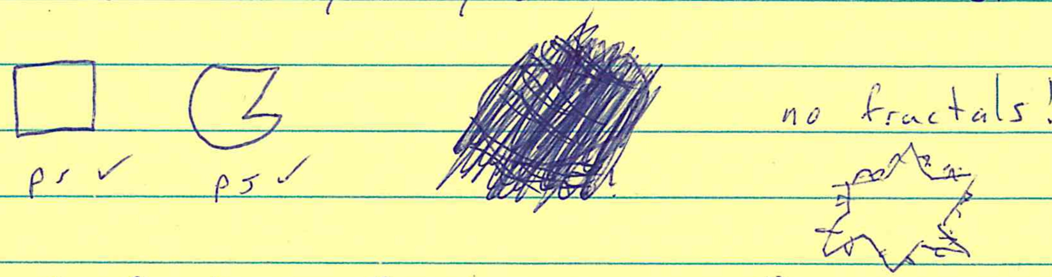
13.4 Green's Thm

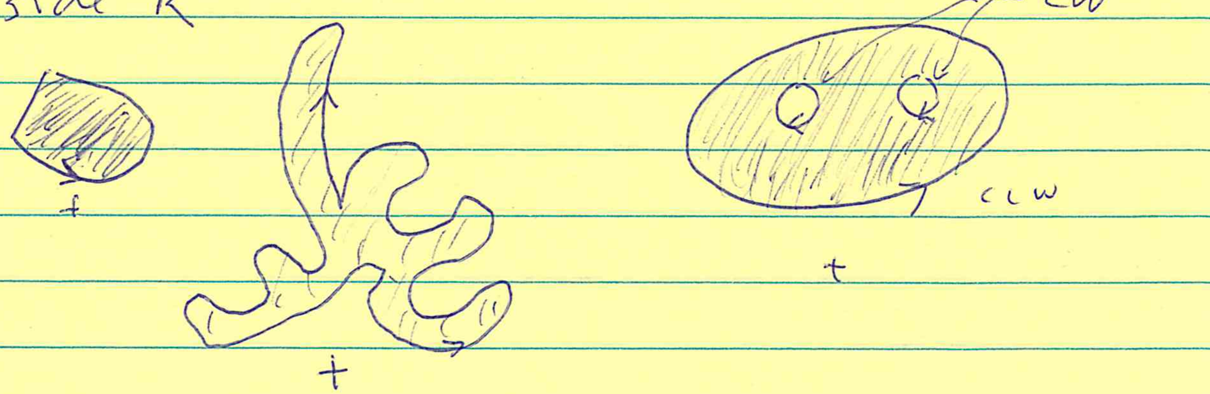
Def A curve in $\mathbb{R}^2$ is a simple closed curve, scc, if it is finite, has no end points, no self intersection points and encloses a region.



A curve is piecewise smooth^{ps} if it is a union of finitely many differentiable curves.



Let R be a finite (bounded) region in $\mathbb{R}^2$ whose boundary is a finite collection of ps scc's: $\partial R = C_1 \cup C_2 \cup \dots C_n$. A curve in ∂R is said to have a positive orientation if when facing with the arrow your left hand would be inside R



Green's Thm Let R be a bounded region of $\mathbb{R}^2$ with $\partial R = C_1 \cup C_2 \cup \cdots \cup C_n$, a finite collection of ps scc, oriented positively. Let $F = \langle M, N \rangle$ be a vector field with continuous partial derivatives in R. Then

$$\oint_{\partial R} F \cdot T \, ds = \iint_R N_x - M_y \, dA$$

This is also written

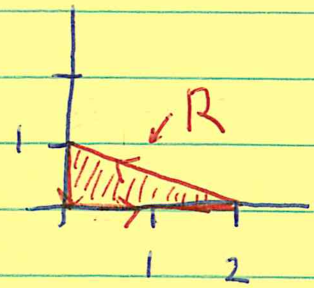
$$\oint_{\partial R} M\,dx + N\,dy = \iint_R N_x - M_y \, dA$$

Rmks If $N_x = M_y$ then $\oint_{\partial R} F \cdot T\,ds = 0$, proving the claim we made about the test for conservative vector fields.

Compare Green's Thm to the FTC and the FTLInt.

There is a 3D version of Green's Thm called Stoke's Thm that we will cover later.

Ex 1



Let $F = \langle -2y + x,\ x+y \rangle$.

Find $\displaystyle\oint_{\partial R} F \cdot T\, ds$.

Sol

We could break up ∂R into 3 line segments and do three line integrals. Instead we will use Green's Thm.

$$\partial_y(-2y+x) = -2 \qquad \partial_x(x+y) = 1.$$

Thus, $\displaystyle\oint_{\partial R} F \cdot T\, ds = \iint_R 1-(-2)\, dA = 3\iint_R dA$

$= 3\ (\text{area of } R) = 3\left(\frac{1}{2}\cdot 2\cdot 1\right) = 3.$

Ex 2 Compute $\displaystyle\oint_c e^x \sin(2y)\,dx + 2e^x\cos(2y)\,dy$, where c is the circle $x^2+y^2=1$, ccw.

Sol The vector field is

$$F\langle e^x \sin(2y),\ 2e^x\cos(2y)\rangle.$$

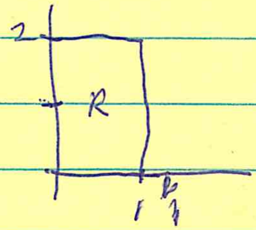
$$\partial_y\left(e^x\sin(2y)\right) = 2e^x\cos(2y)$$

$$\partial_x\left(2e^x\cos(2y)\right) = 2e^x\cos(2y).$$

Thus, $\displaystyle\oint_c F\cdot T\,ds = \iint_{\text{inside } c} 0\ dA = 0.$

Ex 2 Let $F = \langle xy, \frac{1}{3}x^3y + \frac{1}{2}x^2 \rangle$

Let $R = [0,1] \times [0,2]$. Find the work done by F in going around ∂R ccw.



Sol Let $M = xy$ and $N = \frac{1}{3}x^3y + \frac{1}{2}x^2$.

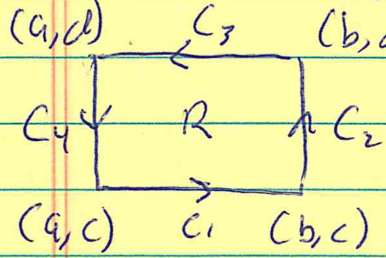
Then $M_y = x$ and $N_x = x^2y + x$.

Thus $N_x - M_y = x^2y$.

$$\oint_{\partial R} F \cdot T\, ds = \iint_R x^2y\, dA = \int_0^2 \int_0^1 x^2y\, dx\, dy = \frac{1}{3}.$$

We will do the proof of Green's Thm where R is a rectangle of the form $[a,b] \times [c,d]$. We will discuss informally this can be used to prove the full version of Green's Thm.

Green's Thm for a rectangle $R = [a,b] \times [c,d]$.



Let $C = C_1 \cup C_2 \cup C_3 \cup C_4 = \partial R$. Let $F = \langle M, N \rangle$. We will show that $\int_C M\,dx + N\,dy = \iint_R N_x - M_y \, dA$.

Let
$$r_1(t) = \langle t, c \rangle \quad a \le t \le b.$$
$$r_2(t) = \langle b, t \rangle \quad c \le t \le d$$
$$r_3(t) = \langle -t, d \rangle \quad -b \le t \le -a$$
$$r_4(t) = \langle a, -t \rangle \quad -d \le t \le -c.$$

$$\oint_C M\,dx = \int_{C_1} M\,dx + \int_{C_2} M\,dx + \int_{C_3} M\,dx + \int_{C_4} M\,dx$$

$$\int_{C_1} M\,dx = \int_a^b M(t,c)\frac{dx}{dt}\,dt = \int_a^b M(t,c)\,dt = \int_a^b M(x,c)\,dx$$

$$\int_{C_2} M\,dx = \int_c^d M(b,t)\frac{dx}{dt}\,dt = \int_c^d 0\,dt = 0.$$

$$\int_{C_3} M\,dx = \int_{-b}^{-a} M(-t,d)\frac{dx}{dt}\,dt = \int_{-b}^{-a} M(-t,d)(-dt) = \star$$

Let $s = -t$. Then $ds = -dt$ and $a \le s \le b$. Thus

$$\star = \int_b^a M(s,d)\,ds = -\int_a^b M(s,d)\,ds = -\int_a^b M(x,d)\,dx.$$

$$\int_{C_4} M\,dx = \int_{-d}^{-c} M(a,-t)\frac{dx}{dt}\,dt = \int_{-d}^{-c} 0\,dt = 0.$$

Thus $\int_C M\,dx = \int_a^b M(x,c) - M(x,d)\,dt$

Next consider, $\iint_R -M_y \, dA$. This equals

$$\int_a^b \int_c^d \frac{-\partial M}{\partial y} \, dy \, dx = -\int_a^b M(x,y)\Big|_{y=c}^{y=d} \, dx$$

$$= -\int_a^b M(x,d) - M(x,c) \, dx =$$

$$= \int_a^b M(x,c) - M(x,d) \, dx = \oint_C M \, dx.$$

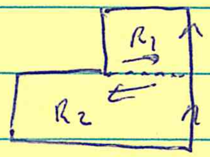
A similar calculation shows $\oint_C N \, dy = \iint_R \frac{\partial N}{\partial x} \, dA$.

Combining these gives Green Thm for rectangles like R.

$$\oint_C M \, dx + N \, dy = \iint_R N_x - M_y \, dA$$

$\boxed{!}$

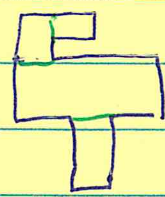
For the general case think about ...



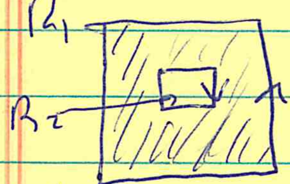
Let $R = R_1 \cup R_2$.

$$\oint_{\partial R} F\cdot T\,ds = \oint_{\partial R_1} F\cdot T\,ds + \oint_{\partial R_2} F\cdot T\,ds.$$

Thus, Green's Thm holds for region made from rectangles



We can cut out ~~region~~ rectangular holes.

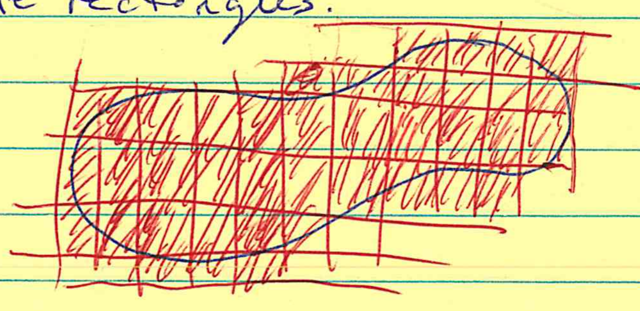


$R = "R_1 - R_2"$

$$\oint_{\partial R} F\cdot T\,ds = \oint_{\substack{\partial R_1\\ ccw}} + \oint_{\substack{\partial R_2\\ cw}} = \oint_{\substack{\partial R_1\\ ccw}} - \oint_{\substack{\partial R_2\\ ccw}}.$$

$$= \iint_{R_1} \_\; dA - \iint_{R_2} \_\; dA = \iint_{R} \_\; dA$$

For smooth regions, we can approximate using little rectangles.



Then take "limit" as the rectangles get smaller and smaller.

Ex Let $F(x,y) = \dfrac{-yi + xj}{x^2+y^2}$. Let C_a be a circle of radius a centered at the origin, oriented ccw. Compute $\oint_{C_a} F\cdot dr$

Old Method Parametrize C_a with $r(t) = \langle a\cos t, a\sin t\rangle$ $0 \le t \le 2\pi$.

$$F = \left\langle \frac{-\cancel{a}\sin t}{a^{\cancel{2}}}, \frac{\cancel{a}\cos t}{a^{\cancel{2}}} \right\rangle$$

$$r'(t) = \langle -a\sin t, a\cos t\rangle$$

$$\int_{C_a} F\cdot dr = \int_0^{2\pi} \left\langle \frac{-\sin t}{a}, \frac{\cos t}{a}\right\rangle \cdot \langle -a\sin t, a\cos t\rangle \, dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} 1 \, dt = 2\pi.$$

Notice that the value of a did not matter.

New Method (Green's Thm) Let $M = \dfrac{-y}{x^2+y^2}$, $N = \dfrac{x}{x^2+y^2}$.

Now $F = \langle M, N \rangle$.

$$M_y = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$N_x = \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}.$$

Thus $M_y = N_x$.

$$\oint_{C_a} F \cdot dr = \iint_{\substack{\text{Disk}\\ \text{inside}\\ C_a}} 0 \, dA = 0.$$

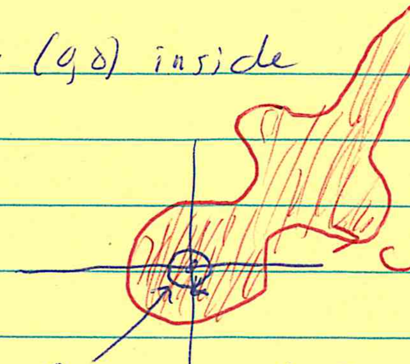
But $0 \neq 2\pi$!! What went wrong?

The problem is F is not defined at $(0,0)$ so Green's Thm does not apply.

But this "mistake" is useful!

Ex Let C be any ps scc that has $(0,0)$ inside the region R it bounds. Then

$$\oint_C F \cdot dr = 2\pi.$$



Pf Let $a>0$ be small enough that C_a is inside R. Orient C ccw and C_a cw. Let R_1 be the region in between C and C_a. In R_1 we can use Green's Thm.

$$\int_{C \cup C_a} F \cdot dr = 0 \quad \text{by Green's Thm.}$$

But

$$0 = \int_{C \cup C_a} F \cdot dr = \oint_{\substack{C \\ ccw}} F \cdot dr + \oint_{\substack{C_a \\ cw}} F \cdot dr = \int_C F \cdot dr - 2\pi.$$

Thus $$\oint_C F \cdot dr = 2\pi.$$

Area Consider a region R with boundary C parametrized by $r(t)$, ccw. The area inside is

$$\text{area} = \iint_R dA$$

If $F = \langle M, N \rangle$ is a vector field s.t. $N_x - M_y = 1$ then

$$\iint_R dA = \oint_C F \cdot T\, ds$$

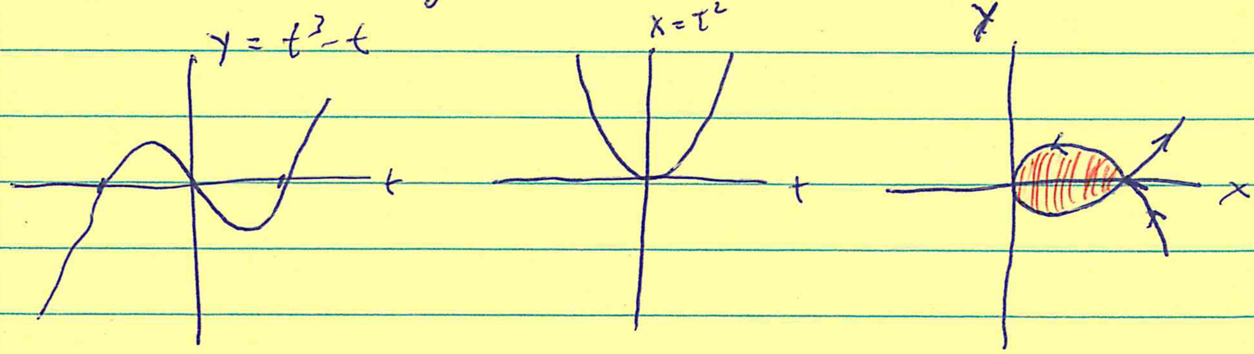
One such v.f. is $N = x$, $M = 0$. Thus

$$\text{area} = \int_a^b \langle 0, x \rangle \cdot \langle x', y' \rangle\, dt$$
$$= \int_a^b x(t) \frac{dy(t)}{dt}\, dt$$

see also pg 510

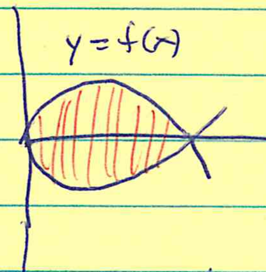
Ex Let $r(t) = \langle t^2, t^3 - t \rangle$, $-1 \le t \le 1$. Find the area of the region enclosed.

Sol



$$A = \int_{-1}^{1} t^2(3t^2 - 1)\, dt = 2\int_0^1 3t^4 - t^2\, dt = 2\left(\frac{3}{5} - \frac{1}{3}\right) = \frac{8}{15}.$$

Note: This example can also be done by "conventional" means. Since $x = t^2$, $t = \sqrt{x}$ and $y = x^{3/2} - x^{1/2}$ for $0 \le x \le 1$.



$$\text{Area} = 2\int_0^1 x^{3/2} - x^{1/2}\,dx = 2\left(\frac{2}{5} - \frac{2}{3}\right) = \frac{8}{15}.$$

Ex Use the same region R but with density $\rho(x,y) = x^2y^2$. Find the ~~center of~~ mass.

Sol ~~$\bar{y} = 0$ by symmetry~~

Mass $= \iint x^2y^2\,dA$ Find $F = \langle M, N\rangle$ s.t.

$N_x - M_y = x^2y^2$. $N = \frac{1}{3}x^3y^2$, $M = 0$.

$$\iint x^2y^2\,dA = \int_{-1}^{1}\left(\frac{1}{3}x^3y^2\right)\left(\frac{dy}{dt}\right)dt$$

$$= \frac{1}{3}\int_{-1}^{1} t^6(t^3-t)^2(3t^2-1)\,dt = \frac{64}{14305} \approx 0.0033$$

$$= \frac{2}{3}\int_0^1 3t^{14} - 7t^{12} + 5t^{10} - t^8\,dt$$

$$= \frac{2}{3}\left(\frac{3}{15} - \frac{7}{13} + \frac{5}{11} - \frac{1}{9}\right) =$$