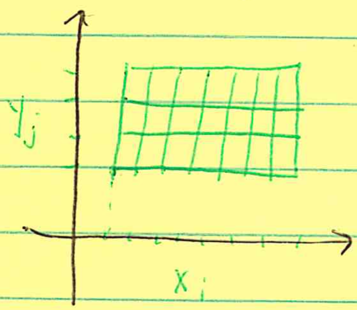
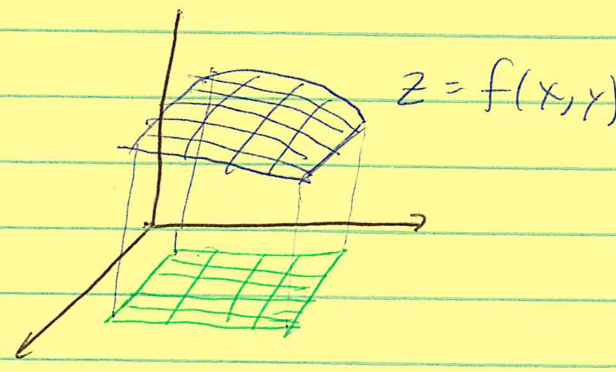
13.6 Surface Area

Let $z = f(x,y)$. Its graph is a surface. We wish to find the surface area of the graph over some region in the xy-plane. For simplicity suppose the region is a rectangle. We partition the region with a grid just as we did with double integrals.





The volume under the graph would be $\iint f(x,y)\,dx\,dy$. To get the surface area we will use an approximation that is the sum of small parallelogram areas that are tangent to the surface. In the limit this will give an area.

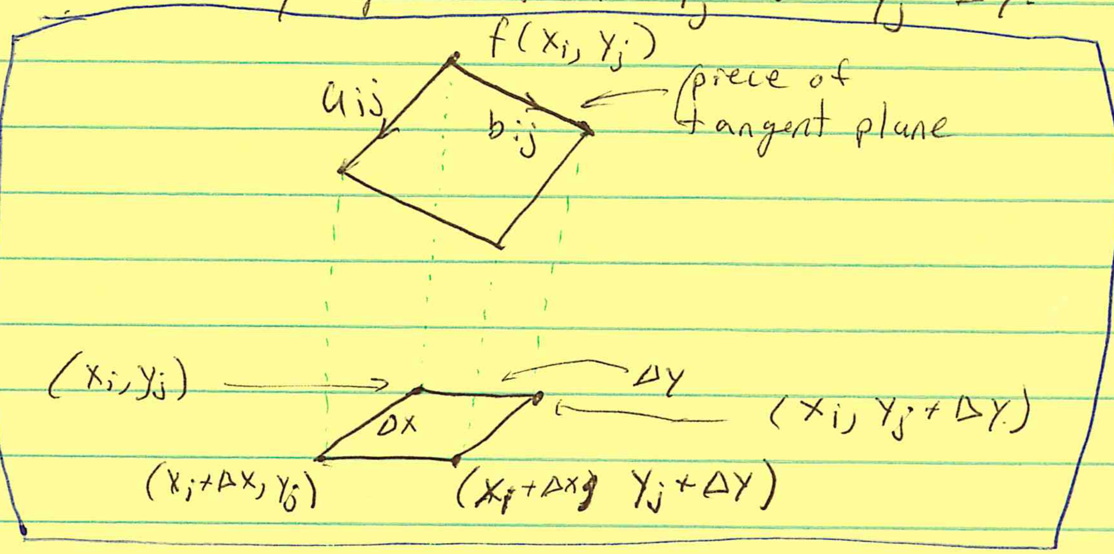
For a given point (x_i, y_j) recall that the vectors

$$\langle 1, 0, f_x(x_i, y_j)\rangle$$
$$\langle 0, 1, f_y(x_i, y_j)\rangle$$

where used to span the plane tangent to the surface at $(x_i, y_j, f(x_i, y_j))$.

We will rescale these vectors so that the parallelogram they determine will have approximately the same area as the patch on the surface given as

x goes from x_i to $x_i + \Delta x$,
and y goes from y_j to $y_j + \Delta y$.



Let $a_{ij} = \langle \Delta x, 0, f_x(x_i, y_j)\Delta x \rangle$,

and $b_{ij} = \langle 0, \Delta y, f_y(x_i, y_j)\Delta y \rangle$.

The area of the parallelogram is $|a_{ij} \times b_{ij}|$.

Then the approximate surface area of $z = f(x, y)$ over the rectangle is

$$S.A. \approx \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij} \times b_{ij}|.$$

We will rewrite this in the form of a Riemann sum (see page 692) so that the limit as $\Delta x, \Delta y \to 0$ gives a double integral.

$$|a_{ij} \times b_{ij}| = |\langle \Delta x, 0, f_x \Delta x \rangle \times \langle 0, \Delta y, f_y \Delta y \rangle|$$

$$= |\langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle| \, \Delta x \Delta y$$

$$= |\langle -f_x, -f_y, 1 \rangle| \, \Delta x \Delta y$$

$$= \sqrt{f_x^2 + f_y^2 + 1} \; \Delta x \Delta y.$$

Now

$$\star = \sum_{i=1}^{n} \sum_{j=1}^{m} \sqrt{(f_x(x_i, y_j))^2 + (f_y(x_i, y_j))^2 + 1} \; \Delta x \Delta y.$$

In the limit as $\Delta x, \Delta y \to 0$ $(n, m \to \infty)$ we get,

$$S.A. = \iint_{\text{Region}} \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1} \; dx\,dy$$

or $dy\,dx$

or $r\,dr\,d\theta$.

((Compare this with arc length formula for $y = f(x)$, $a \le x \le b$: $\int_a^b \sqrt{(f'(x))^2 + 1} \; dx$.)

Ex Let $r(u,v) = \langle u+v,\ uv,\ u-v \rangle$. Find the surface area for the surface given by $0 \le u \le 1$, $0 \le v \le 1$.

Sol $r_u = \langle 1, v, 1 \rangle$, $r_v = \langle 1, u, -1 \rangle$.

$r_u \times r_v = \langle -v-u,\ 2,\ u-v \rangle$.

$$|r_u \times r_v| = \left[(-v-u)^2 + 4 + (u-v)^2\right]^{1/2}$$

$$= \left[v^2 + 2vu + u^2 + 4 + u^2 - 2uv + v^2\right]^{1/2}$$

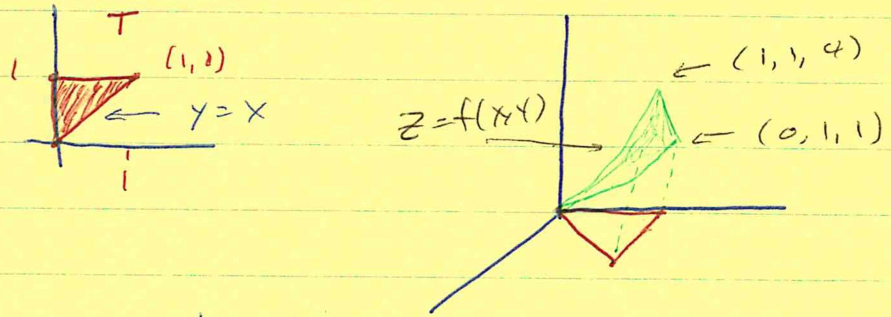
$$= \left[2u^2 + 2v^2 + 4\right]^{1/2}.$$

Thus, $$S.A. = \int_0^1 \int_0^1 \sqrt{2u^2 + 2v^2 + 4}\ du\,dv$$

$$\approx \underline{2.302310960}$$ (done on Maple).

Ex Let T be the triangular region in the xy-plane with vertice $(0,0)$, $(0,1)$ and $(1,1)$. Find the surface area of $z = f(x,y) = 3x + y^2$ over T.

Sol.



$$S.A. = \int_0^1 \int_0^y \sqrt{(f_x)^2 + (f_y)^2 + 1} \, dx \, dy$$

$f_x = 3$, $f_y = 2y$. Thus,

$$S.A. = \int_0^1 \int_0^y \sqrt{4y^2 + 10} \, dx \, dy$$

(Doing $dy\,dx$ would be harder.)

$$= \int_0^1 y\sqrt{4y^2+10} \, dy$$

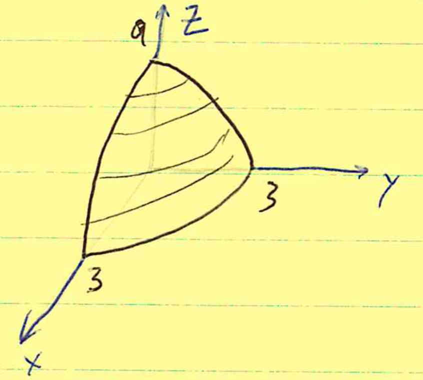
Let $u = 4y^2 + 10$. Then $du = 8y\,dy$ and $10 \le u \le 14$.

$$S.A. = \frac{1}{8} \int_{10}^{14} \sqrt{u} \, du = \frac{1}{8} \frac{2}{3} u^{3/2} \Big|_{10}^{14} = \frac{1}{12}\left(14^{3/2} - 10^{3/2}\right)$$

$$= \frac{14\sqrt{14} - 10\sqrt{10}}{12} \approx \underline{1.730035568}$$

Ex Find the surface area of the part of the paraboliod $z = f(x,y) = 9 - x^2 - y^2$ that lies above the xy-plane.

Sol



$$S.A. = \underset{\substack{\text{disk of} \\ \text{radius } 3}}{\iint} \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA.$$

We will use polar coordinates.

$f_x = -2x$, $f_y = -2y$ so $\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4r^2 + 1}$

Thus

$$S.A. = \int_0^{2\pi} \int_0^3 \sqrt{4r^2+1}\, r\,dr\,d\theta = 2\pi \int_0^3 \sqrt{4r^2+1}\, r\,dr$$

Let $u = 4r^2 + 1$. Then $du = 8r\,dr$, $1 \le u \le 37$.

$$S.A. = 2\pi \cdot \frac{1}{8} \int_1^{37} u^{1/2}\, du = \frac{\pi}{4} \cdot \frac{2}{3} \cdot u^{3/2} \Big|_1^{37}$$

$$= \frac{\pi}{6}\left(37^{3/2} - 1\right) = \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$

$$\approx \underline{117.3187007.}$$

Parametric Surfaces

Ex Let $\mathbf{r}(\theta, \phi) = \langle R\cos\theta\sin\phi,\ R\sin\theta\sin\phi,\ R\cos\phi \rangle$.

$$\underbrace{R\cos\theta\sin\phi}_{x(\theta,\phi)} \quad \underbrace{R\sin\theta\sin\phi}_{y(\theta,\phi)} \quad \underbrace{R\cos\phi}_{z(\theta,\phi)}$$

for $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$. This is a parametric equation for the sphere

$$x^2 + y^2 + z^2 = R^2.$$

More generally, a papametric surface is given by

$$\mathbf{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$

for (u,v) is some domain $D \subset \mathbb{R}^2$.

Ex Express the plane $2x - y + z = 5$ in terms of parameters (x,y) and (x,z).

(x,y): $z = 5 + y - 2x$. Let $\mathbf{r}(x,y) = \langle x, y, 5 + y - 2x \rangle$.

(x,z): $y = 2x + z - 5$. Let $\mathbf{r}(x,z) = \langle x, 2x + z - 5, z \rangle$.

Ex Express the paraboloid $z = 9 - x^2 - y^2$ in terms of parameters (x,y) and (r,θ).

(x,y) $\mathbf{r}(x,y) = \langle x, y, 9 - x^2 - y^2 \rangle$.

(r,θ) $\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 9 - r^2 \rangle$ (Don't confuse $\mathbf{r}$ the vector with r the variable.)

Surface Area of a Parametric Surface

We know for $z = f(x,y)$, $S.A. = \iint \sqrt{f_x^2 + f_y^2 + 1}\, dx\,dy$.

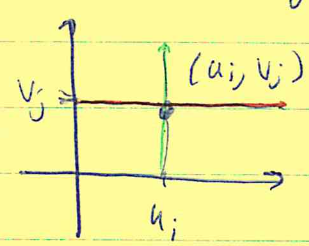
To generalize this we consider the tangent plane to a parametric surface. Let

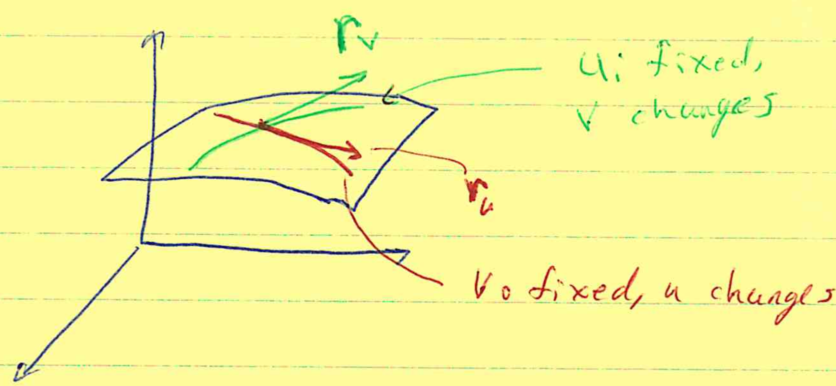
$$r(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle.$$

Let (u_i, v_j) be a given point in the uv-plane. Consider $r(u_i, v) = \langle x(u_i, v), y(u_i, v), z(u_i, v) \rangle$. As v varries with u_i fixed $r(u_i, v)$ traces out a curve in our surface. The vector

$$r_v = \frac{\partial r}{\partial v}(u_i, v_j)$$

will be tangent to this curve at the point $r(u_i, v_j)$.





Likewise we can hold v_j fixed while u varries. The $r_u(u_i, v_j)$ is tangent to the curve $r(u, v_j)$.

Together $r_u(u_i, v_j)$ and $r_v(u_i, v_j)$ span the tangent plane to our surface at the point $(x(u_i, v_j), y(u_i, v_j), z(u_i, v_j))$.

<u>Ex</u> Let $r(u,v) = \langle u^2+v, uv+2, u^2+v^2 \rangle$. Find the tangent plane when $u=1, v=2$.

<u>Sol</u> $r(1,2) = \langle 3, 4, 5 \rangle$ will be the point of tangency.

$r_u = \langle 2u, v, 2u \rangle = \langle 2, 2, 2 \rangle$.
$r_v = \langle 1, u, 2v \rangle = \langle 1, 1, 4 \rangle$.

For the normal vector let $n = \langle 2,2,2 \rangle \times \langle 1,1,4 \rangle = \langle 6,-6,0 \rangle$.
(Check this!)

For simplicity we use $n = \langle 1,-1,0 \rangle$ instead.

~~The~~ An equation of the tangent plane is

$$\langle 1,-1,0 \rangle \cdot \langle x-3, y-4, z-5 \rangle = 0$$

or $$x - y = -1.$$

Now, to apply this to surface area we again scale r_u and r_v so that the parallelogram they determine has approximately the same area as a patch on the surface determined by

$$u_i \le u \le u_i + \Delta u,$$
$$v_j \le v \le v_j + \Delta v.$$

Let $\Delta A_{ij} = |(\Delta u\, r_u(u_i, v_j)) \times (\Delta v\, r_v(u_i, v_j))|$

$$= |r_u(u_i, v_j) \times r_v(u_i, v_j)|\, \Delta u \Delta v.$$

Then $S.A. \approx \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta A_{ij} = \sum \sum |r_u \times r_v|\, \Delta u \Delta v.$

In the limit we get

$$S.A. = \iint \underbrace{|r_u \times r_v|\, du\, dv}_{= dS}.$$

Ex Let $r(\alpha,\beta) = \langle \sin(\pi\alpha\beta), \cos(\pi\alpha\beta), \alpha+\beta \rangle$ for $0 \le \alpha \le 1$, $0 \le \beta \le 1$. Find surface area.

Sol $r_\alpha = \langle \pi\beta\cos(\pi\alpha\beta), -\pi\beta\sin(\pi\alpha\beta), 1 \rangle.$

$r_\beta = \langle \pi\alpha\cos(\pi\alpha\beta), -\pi\alpha\sin(\pi\alpha\beta), 1 \rangle.$

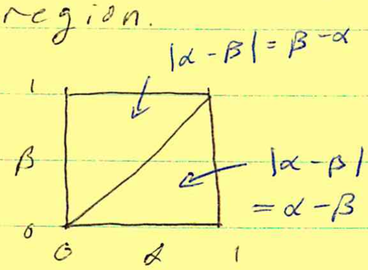
$r_\alpha \times r_\beta = \langle -\pi\beta\sin(\pi\alpha\beta) + \pi\alpha\sin(\pi\alpha\beta), \pi\alpha\cos(\pi\alpha\beta) - \pi\beta\cos(\pi\alpha\beta), 0 \rangle.$

$= \langle \pi(\alpha-\beta)\sin(\pi\alpha\beta), \pi(\alpha-\beta)\cos(\pi\alpha\beta), 0 \rangle.$

$|r_\alpha \times r_\beta|^2 = \pi^2(\alpha-\beta)^2\sin^2(\pi\alpha\beta) + \pi^2(\alpha-\beta)^2\cos^2(\pi\alpha\beta)$

$= \pi^2(\alpha-\beta)^2.$

$|r_\alpha \times r_\beta| = \pi|\alpha-\beta|.$ We will need to split up the region.

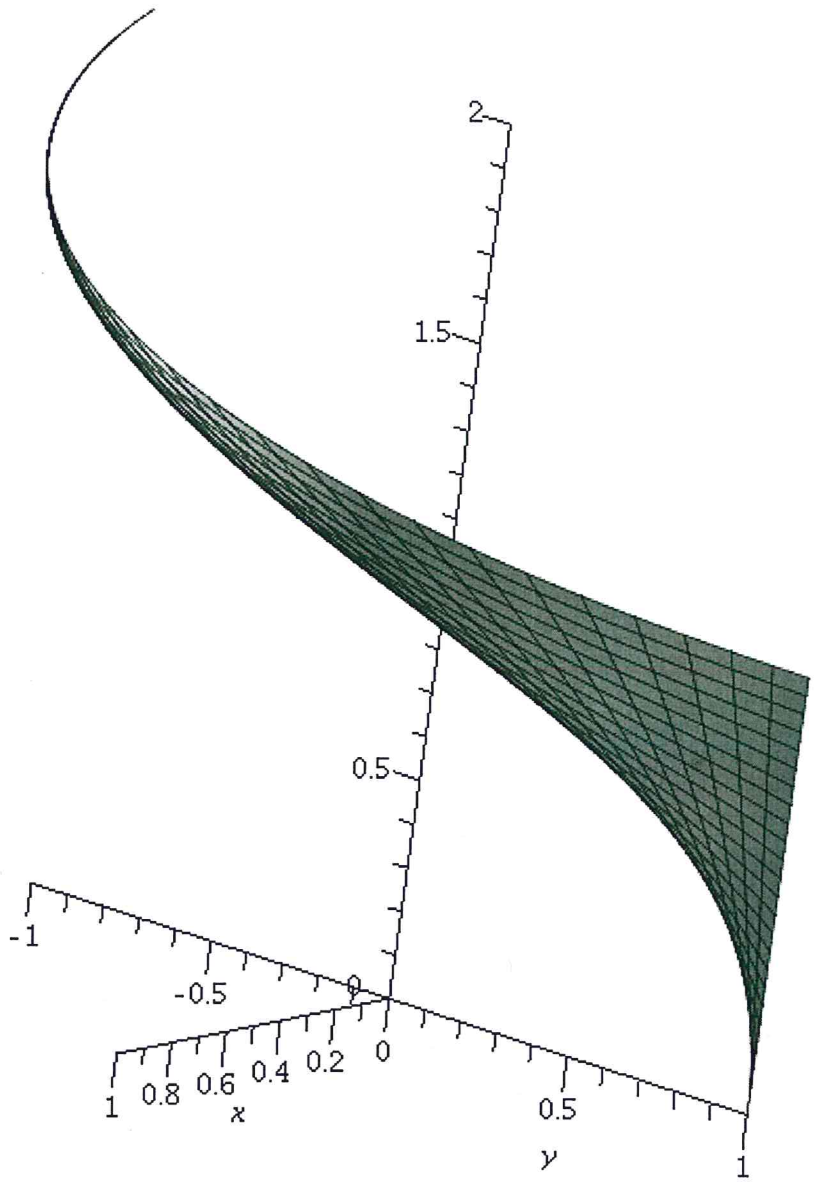


$S.A. = \int_0^1\int_0^1 \pi|\alpha-\beta|\,d\alpha\,d\beta =$

$\int_0^1\int_\beta^1 \pi(\alpha-\beta)\,d\alpha\,d\beta + \int_0^1\int_0^\beta \pi(\beta-\alpha)\,d\alpha\,d\beta$

$= \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}.$

This is the surface from the last example.



Area $= \frac{\pi}{3}$.

Example 9 on page 803 of the textbook is very important. Study it carefully, working out all the details. It show that the standard (θ, ϕ) parametrization of a sphere of radius R that

$$|r_\theta \times r_\phi| = R^2 \sin\phi.$$

Ex Use this to find the formula for the surface area sphere of radius R.

$$\int_0^{2\pi}\int_0^{\pi} R^2 \sin\phi \, d\phi \, d\theta = 4\pi R^2.$$

Ex The electric charge density on a hemisphere of radius R is proportional to the distance to the base disk. Find the total charge.

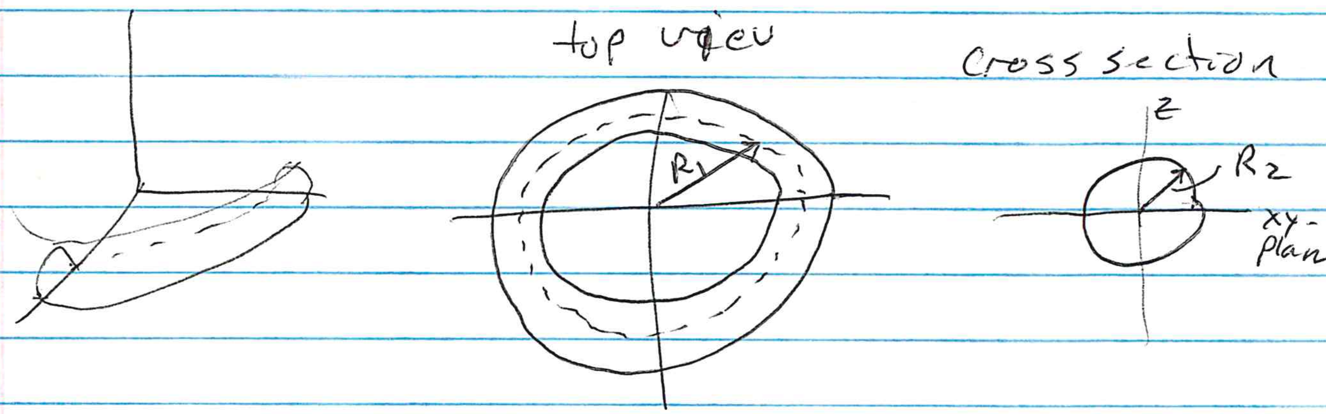
Sol.

$$\int_0^{2\pi}\int_0^{\pi/2} \underbrace{(kz)}_{\text{density}} (R^2 \sin\phi) \, d\phi \, d\theta$$

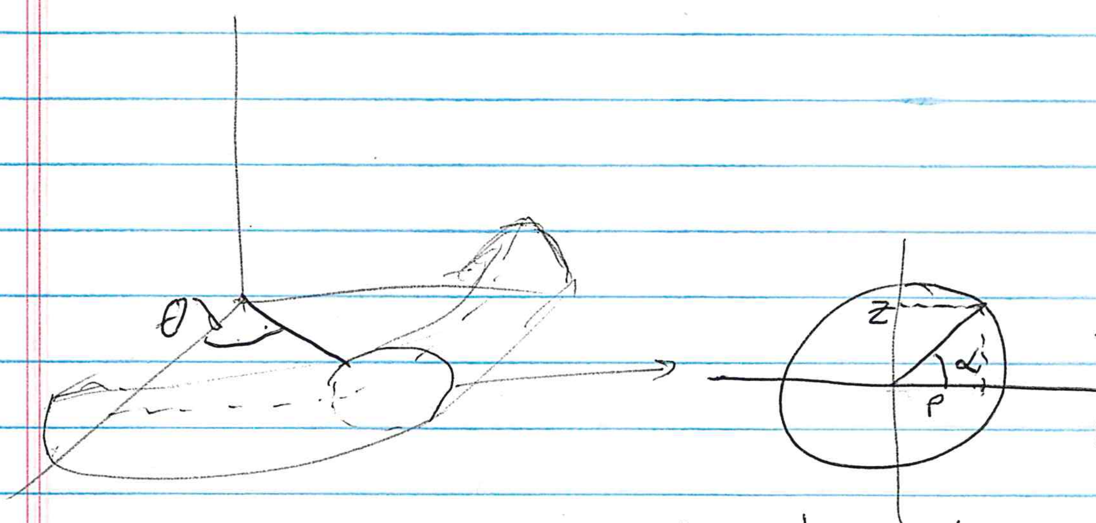
$$= 2\pi k R^3 \int_0^{\pi/2} \cos\phi \sin\phi \, d\phi$$

$$= \pi k R^3. \quad (\text{check this.})$$

Ex Find a formula for the surface area of a torus where R_1 is the radius of the core circle and R_2 is the radius of a cross sectional circle. See figure



Sol. First we derivate parametric equations for the torus, in terms of θ and α.



$z = R_2 \sin\alpha$

$p = R_2 \cos\alpha$.

Thus the polar $r = R_1 + R_2 \cos\alpha$.

Thus,

$$x = r\cos\theta = (R_1 + R_2\cos\alpha)\cos\theta$$
$$y = r\sin\theta = (R_1 + R_2\cos\alpha)\sin\theta$$
$$z = R_2\sin\alpha.$$

$0 \le \alpha \le 2\pi$

$0 \le \theta \le 2\pi$

Next computer r_θ and r_α, where $r = \langle x(\theta,\alpha), y(\theta,\alpha), z(\theta,\alpha)\rangle$

r_θ: $\dfrac{\partial x}{\partial \theta} = -(R_1 + R_2 \cos\alpha)\sin\theta$

$$\frac{\partial y}{\partial \theta} = (R_1 + R_2 \cos\alpha)\cos\theta$$

$$\frac{\partial z}{\partial \theta} = 0$$

r_α: $\dfrac{\partial x}{\partial \alpha} = -R_2 \sin\alpha \cos\theta$

$$\frac{\partial y}{\partial \alpha} = -R_2 \sin\alpha \sin\theta$$

$$\frac{\partial z}{\partial \alpha} = R_2 \cos\alpha$$

Now $r_\theta \times r_\alpha = \langle y_\theta z_\alpha - \cancel{z_\theta y_\alpha}^{\,0},\ \cancel{z_\theta x_\alpha}^{\,0} - x_\theta z_\alpha,\ x_\theta y_\alpha - y_\theta x_\alpha \rangle$

$$|r_\theta \times r_\alpha|^2 = y_\theta^2 z_\alpha^2 + x_\theta^2 z_\alpha^2 + (x_\theta y_\alpha - y_\theta x_\alpha)^2$$

$$= (y_\theta^2 + x_\theta^2) z_\alpha^2 + (\qquad\qquad)^2$$

$$= (R_1 + R_2\cos\alpha)^2 R_2^2 \cos^2\alpha + \left(R_2(R_1+R_2\cos\alpha)\sin\alpha\sin^2\theta + R_2(R_1+R_2\cos\alpha)\sin\alpha\cos^2\theta\right)^2$$

$$= (R_1+R_2\cos\alpha)^2 R_2^2 \cos^2\alpha + (R_1+R_2\cos\alpha)^2 R_2^2 \sin^2\alpha$$

$$= (R_1+R_2\cos\alpha)^2 R_2^2.$$

Finally, $S.A. = \int_0^{2\pi}\int_0^{2\pi} |r_\theta \times r_\alpha| \, d\theta \, d\alpha$

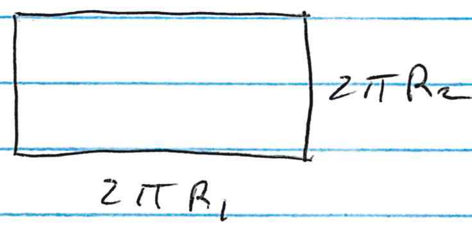
$$= \int_0^{2\pi}\int_0^{2\pi} (R_1 + R_2 \cos\alpha) R_2 \, d\theta \, d\alpha$$

$$= R_2(2\pi) \cdot \int_0^{2\pi} R_1 + R_2 \cancelto{0}{\cos\alpha} \, d\alpha$$

$$= R_2 (2\pi) R_1 (2\pi) = 4\pi^2 R_1 R_2$$

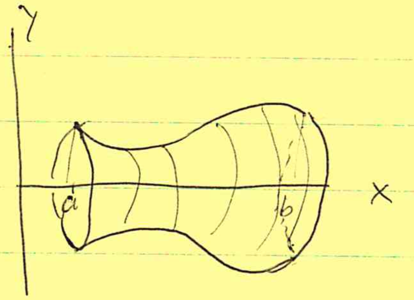
Compare to:

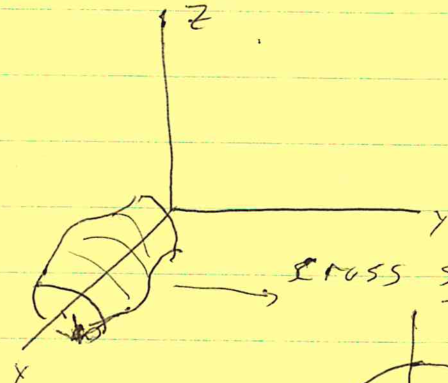
fold out torus $\longrightarrow$



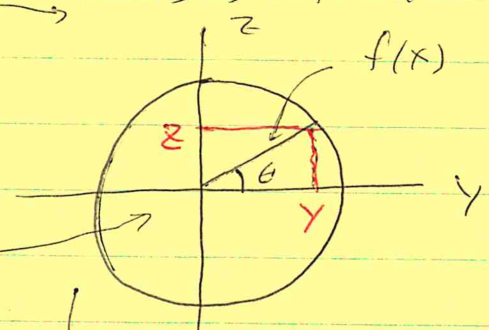
Surfaces of Revolution

Let $y = f(x)$, where $f(x) \geq 0$ on $[a,b]$. We will find a formula for the surface area of the surface formed by rotating $y = f(x)$ about the x-axis.





cross sector at some x_0



We will parameterize the surface using x and θ as shown. Then

$$y = f(x)\cos\theta$$
$$z = f(x)\sin\theta$$

$$r(x,\theta) = \langle x(x,\theta), y(x,\theta), z(x,\theta)\rangle$$

where,

$$x = x, \quad y = f(x)\cos\theta, \quad z = f(x)\sin\theta.$$

Thus,

$$r_x = \frac{\partial r}{\partial x} = \langle 1, f'(x)\cos\theta, f'(x)\sin\theta\rangle$$

$$r_\theta = \frac{\partial r}{\partial \theta} = \langle 0, -f(x)\sin\theta, f(x)\cos\theta\rangle.$$

$$r_x \times r_\theta = \langle f'f\cos^2\theta + f'f\sin^2\theta, -f\cos\theta, -f\sin\theta\rangle$$

$$= \langle f'f, -f\cos\theta, -f\sin\theta\rangle.$$

$$|r_x \times r_\theta| = \sqrt{(f'f)^2 + f^2\cos^2\theta + f^2\sin^2\theta}$$

$$= \sqrt{(f'f)^2 + f^2}$$

$$= f\sqrt{(f')^2 + 1}$$

Thus,

$$S.A. = \int_0^{2\pi}\int_a^b f(x)\sqrt{[f'(x)]^2 + 1}\ dx\,d\theta$$

$$= 2\pi\int_a^b f(x)\sqrt{[f'(x)]^2 + 1}\ dx$$

Notice $2\pi f(x)$ is the circumference of the circle at x and $\sqrt{(f'(x))^2+1}\,dx = ds$ = arc length differential.

Compare to section 7.5, pg 395.

Next we do examples.

Ex Find the S.A. when $y = \sqrt{x}$, $0 \le x \le 1$, is rotated about the x-axis.

Sol Let $f(x) = \sqrt{x} = x^{\frac{1}{2}}$, $f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$, $[f'(x)]^2 = \frac{1}{4x}$.

Thus,

$$S.A. = 2\pi \int_0^1 \sqrt{x} \sqrt{\frac{1}{4x} + 1} \, dx$$

$$= 2\pi \int_0^1 \sqrt{\frac{1}{4} + x} \, dx.$$

Let $u = \frac{1}{4} + x$. Then $du = dx$. $\frac{1}{4} \le u \le \frac{5}{4}$.

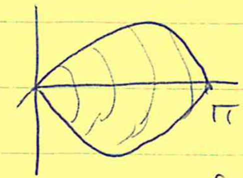
$$S.A. = 2\pi \int_{\frac{1}{4}}^{\frac{5}{4}} u^{\frac{1}{2}} du = 2\pi \, \frac{2}{3} u^{\frac{3}{2}} \Big|_{\frac{1}{4}}^{\frac{5}{4}}$$

$$= \frac{4\pi}{3}\left(\left(\frac{5}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right) = \frac{4\pi}{3} \, \frac{5^{3/2} - 1^{3/2}}{8}$$

$$= \frac{\pi}{6}(5\sqrt{5} - 1) \approx 5.3304135$$

Ex Find the S.A. of the surface formed by rotating $y = \sin x$ over $[0, \pi]$ about the x-axis.

Sol $S.A. = 2\pi \int_0^{\pi} \sin x \sqrt{\cos^2 x + 1}\, dx$



Let $u = \cos x$. Then $du = -\sin x$. u goes from 1 to -1.

$$S.A. = -2\pi \int_1^{-1} \sqrt{u^2+1}\, du = 2\pi \int_{-1}^{1} \sqrt{u^2+1}\, du = 4\pi \int_0^1 \sqrt{u^2+1}\, du.$$

Let $u = \tan w$. Then $du = \sec^2 w\, dw$ and w goes from 0 to $\pi/4$. Thus,

$$S.A. = 4\pi \int_0^{\pi/4} \sqrt{\tan^2 w + 1}\, \sec^2 w\, dw = 4\pi \int_0^{\pi/4} \sec^3 w\, du.$$

See page 321, Example 8 in textbook. It gives

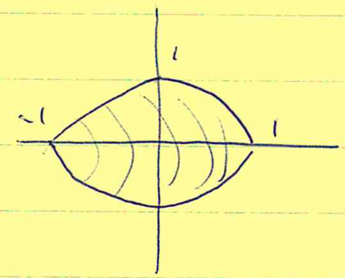
$$S.A. = 4\pi \cdot \frac{1}{2} \Big[\sec w + \tan w + \ln|\sec w + \tan w| \Big]_0^{\pi/4}$$

$$= 2\pi \Big[\big[\sqrt{2} + 1 + \ln(\sqrt{2}+1) \big] - \big[1 + 0 + \ln|1+0| \big] \Big]$$

$$2\pi \left[\sqrt{2} + \ln(\sqrt{2}+1) \right] \approx 14.42359945.$$

Ex Find the surface area of the surface generated by rotating the curve $y = 1 - x^2$, $-1 \le x \le 1$, about the x-axis.

Sol



$$S.A. = 2\pi \int_{-1}^{1} f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

Now $f(x) = 1 - x^2$. $f'(x) = -2x$. Thus,

$$S.A. = 2\pi \int_{-1}^{1} (1 - x^2) \sqrt{1 + 4x^2}\, dx$$

$$= 4\pi \int_0^1 (1 - x^2) \sqrt{1 + 4x^2}\, dx$$

Computer integration gives

$$S.A. = 4\pi \left(\frac{7\sqrt{5}}{16} + \frac{17}{32} \ln(2 + \sqrt{5}) \right) \approx 21.93096931$$

To do this integral by hand use $x = \frac{1}{2} \tan\theta$ to get

$$S.A. = \pi \int_0^{\arctan 2} 5\sec^3\theta - \sec^5\theta \, d\theta.$$

Have fun!