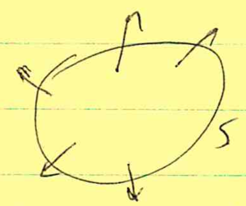
13.8 Stokes' Thm (or the curl theorem)

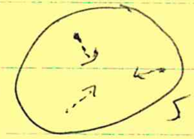
Thm Let S be a piecewise smooth surface in $\mathbb{R}^3$ with boundary. Assume S is oriented and that the direction of C follows the RHR. Let F be a vector field that has continuous partial derivatives on an open region containing S. Then

$$\oint_C F \cdot T \, ds = \iint_S (\text{curl } F) \cdot n \, dS$$

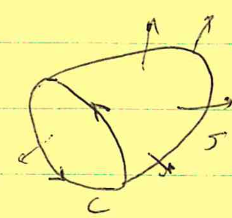
Remarks ① A surface in oriented by choosing a consistant normal direction at each interior pt.



outward orientation

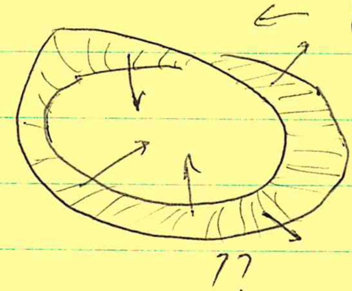


inward.

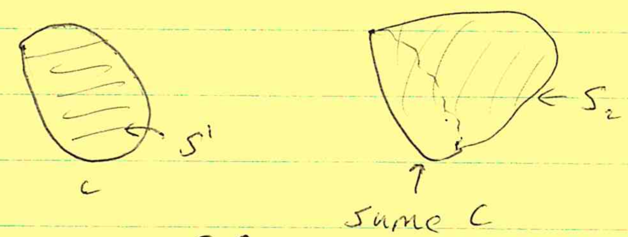


Some surfaces do not allow this. They are said to be non-orientable. The famous Mobiüs band is an example

← half twist.



② Stokes' Thm implies the surface integral of $(\text{curl} F)\cdot n$ depends only on the boundary curve (or curves). So we have "surface independence."



$$\iint_{S_1} (\text{curl} F)\cdot n \, dS = \iint_{S_2} (\text{curl} F)\cdot n \, dS$$

③ If C and S are in $\mathbb{R}^2$ then Stokes' Thm becomes Green's Thm.

Let $F = \langle P, Q, R\rangle$. If $S \subset \mathbb{R}^2$ use $n = \langle 0,0,1\rangle$. C ccw

$$\text{Then } \oint_C F\cdot T \, ds = \iint_S (\text{curl } F)\cdot\langle 0,0,1\rangle \, dS \qquad dS = dA = dx\,dy$$

$$= \iint_S Q_x - P_y \, dx\,dy.$$

④ $\nabla\times F = 0 \Rightarrow \oint_C F\cdot T\,ds = 0$ for any C. (Just find a surface, orientable, with boundary C.)

This is path independence. We already know

$$\text{Path indep.} \Rightarrow F = \nabla f \text{ for some } f.$$

and $\nabla\times(\nabla f) = 0 \Rightarrow \nabla\times F = 0$.

So,

$$\nabla\times F \Leftrightarrow \text{Path independence}$$

Ex 1 $F(x,y,z) = \langle xy, yz, xz \rangle$.

S = unit disk in xy-plane $\{(x,y,0) \mid x^2+y^2 \leq 1\}$.

$C = \partial S$ = unit circle $\{(x,y,0) \mid x^2+y^2 = 1\}$.

Find $\oint_C F \cdot T \, ds$.

Sol $\oint F \cdot T \, ds = \iint (\nabla \times F) \cdot n \, dS$.

$n = \langle 0, 0, 1 \rangle$, $\quad \nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy & yz & xz \end{vmatrix} = \langle -y, -z, -x \rangle$.

Thus, $(\nabla \times F) \cdot n = -x$.

We use polar coordinates. $dS = r\,dr\,d\theta$

$$\int_0^{2\pi} \int_0^1 -(r\cos\theta)\, r\,dr\,d\theta = 0.$$

Ex 2 Let S be the graph of $z = 9 - x^2 - y^2$ for $z \geq 0$.
Let $C = \partial S$: $x^2 + y^2 = 9$, $z = 0$.
Let $F = \langle 3z, 4x, 2y \rangle$.

Find $\oint_C F \cdot T\, ds$ 3 different ways!

Sol I (old way). Let $r(t) = \langle 3\cos t, 3\sin t, 0 \rangle$, $0 \leq t \leq 2\pi$.

$$\oint_C F \cdot T\, ds = \int_0^{2\pi} F \cdot \frac{dr}{dt}\, dt$$

$$\frac{dr}{dt} = \langle -3\sin t, 3\cos t, 0 \rangle$$

$$F = \langle 3 \cdot 0, 4 \cdot 3\cos t, 2 \cdot 3\sin t \rangle$$
$$= \langle 0, 12\cos t, 6\sin t \rangle$$

$$F \cdot \frac{dr}{dt} = 36\cos^2 t$$

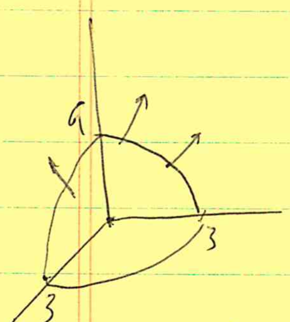
$$\int_0^{2\pi} 36\cos^2 t\, dt = \boxed{36\pi}$$

Sol II By Stokes' Thm $\oint_C F\cdot T\,ds = \iint_S (\nabla\times F)\cdot n\, dS.$

$\nabla\times F = \langle 2,3,4\rangle$

Find n. Let $g(x,y,z) = z + x^2+y^2$. Then $g=9$ gives our surface.

$$\nabla g = \langle 2x, 2y, 1\rangle \qquad n = \frac{\pm\nabla g}{|\nabla g|}.$$



From picture use $+$.

$$dS = |\nabla g|\, dA.$$

$$\iint (\nabla\times F)\cdot\frac{\nabla g}{|\nabla g|}\,|\nabla g|\,dA = \iint \langle 2,3,4\rangle\cdot\langle 2x,2y,1\rangle\, dA$$

$$= \iint 4x+6y+4\,dA = \int_0^{2\pi}\int_0^3 (\cancelto{0}{4r\cos\theta} + \cancelto{0}{6r\sin\theta}+4)\,r\,dr\,d\theta$$

$$= 8\pi\int_0^3 r\,dr = 8\pi\cdot\frac{9}{2} = 36\pi$$

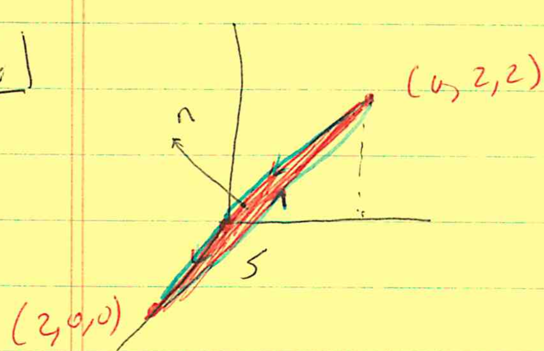
Sol III Replace S with the disk of radius 3 in the xy-plane, center $(0,0,0)$. Now $n = \langle 0,0,1\rangle$.

$$(\nabla\times F)\cdot n = \langle 2,3,4\rangle\cdot\langle 0,0,1\rangle = 4$$

$$\int_0^{2\pi}\int_0^3 4\,r\,dr\,d\theta = 36\pi$$

Ex3 Let C be the boundary of the triangle with vertices
$(0,0,0)$, $(2,0,0)$, $(0,2,2)$
oriented ccw when viewed from above.
Let $F = \langle y, z, x \rangle$. Find $\oint_C F \cdot T\, ds$.

Sol



By Stokes' Thm $\oint_C F \cdot T\, ds = \iint_S (\text{curl } F) \cdot n\, dS$.

$\nabla \times F = \langle -1, -1, -1 \rangle$.

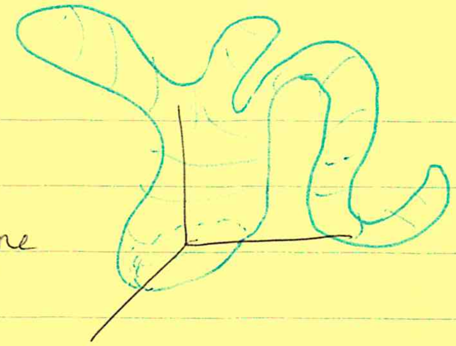
Find n. $n = \dfrac{\pm \langle 2,0,0 \rangle \times \langle 0,2,2 \rangle}{|\quad|} = \dfrac{\pm \langle 0,-4,4 \rangle}{\sqrt{32}}$

Use $n = \langle 0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

$(\nabla \times F) \cdot n = \langle -1,-1,-1 \rangle \cdot \langle 0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = 0$.

Hence $\oint_C F \cdot T\, ds = \iint 0\, dS = 0$.

What if we had gotten $(\nabla \times F) \cdot n = 7$. Then the answer would by $7 \times$ the area of the triangle $= 7 \cdot 2\sqrt{2} = 14\sqrt{2}$.



Ex 4 Let $F = \langle 3y, -2x, xyz \rangle$.
Let S be as in the figure
with $\partial S = C =$ unit circle in xy-plane

Find $\iint_S (\nabla \times F) \cdot n \, dS$.

Sol By Stokes' Thm we can replace S with the unit disk in the xy-plane

$$\iint_{disk} (\nabla \times F) \cdot \langle 0, 0, 1 \rangle \, dA$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 3y & -2x & xyz \end{vmatrix} = \langle \text{don't care}, \text{don't care}, -5 \rangle$$

$$\iint_{\substack{unit \\ disk}} -5 \, dA = -5\pi.$$

Ex 5 Let S and $C = \partial S$ satisfy the conditions of Stokes' Thm. Let f and g be scalar functions with all second order ~~partial~~ partial derivatives continuous. Prove that

$$\oint_C (f \nabla g) \cdot T \, ds = \iint_S (\nabla f \times \nabla g) \cdot n \, dS$$

Proof By Stokes' Thm

$$\oint_C (f \nabla g) \cdot T \, ds = \iint_S \nabla \times (f \nabla g) \cdot n \, dS.$$

But $\nabla \times (f \nabla g) = f(\nabla \times \nabla g) + \nabla f \times \nabla g$ by #26 in 13.5.

And $\nabla \times \nabla g = 0$ by Thm 3 in 13.5.

Thus,

$$\oint_C (f \nabla g) \cdot T \, ds = \iint_S (\nabla f \times \nabla g) \cdot n \, dS$$

as claimed.

Concluding Remarks

We have $\nabla \times F = 0 \iff F = \nabla f$ for some f

$\iff$ path independence

It turns out $\nabla \cdot F \iff F = \nabla \times G$ for some vector field G.

If $F = \nabla f$, then f is called a potential function for F.

If $F = \nabla \times G$, then G is called a vector potential for F. (This works for magnetic fields.)