Proof that $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

<u>Def</u> A <u>spherical wedge</u> is a region given by

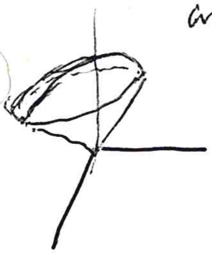
$$\{(\rho,\theta,\phi) \mid \rho_1 \le \rho \le \rho_2,\ \theta_1 \le \theta \le \theta_2,\ \phi_1 \le \phi \le \phi_2\}$$

where $0 \le \rho_1 < \rho_2$, $0 \le \theta_1 < \theta_2 \le 2\pi$, $0 \le \phi_1 < \phi_2 \le \pi$.

<u>Theorem</u> The volume of a spherical wedge is given by

$$\frac{\rho_2^3 - \rho_1^3}{3}(\cos\phi_1 - \cos\phi_2)(\theta_2 - \theta_1)$$

<u>Proof</u> <u>Step 1</u> We compute the volume inside the cone along the positive z-axis given a fixed ϕ, and inside the sphere of radius ρ.



(Think ice cream cone)

disk method

$$\text{cone} + \text{cap} = \tfrac{1}{3}\,\text{base} \times \text{height} + \int \pi r^2 \, dz$$

$$= \tfrac{1}{3}\pi(\rho\sin\phi)^2 \rho\cos\phi + \pi \int_{\rho\cos\phi}^{\rho} \rho^2 - z^2 \, dz$$

$$= \frac{\pi}{3}\rho^3 \sin^2\phi\cos\phi + \pi\left(\rho^2 z - \tfrac{1}{3}z^3\right)\Big|_{\rho\cos\phi}^{\rho}$$

$$= \quad '' \quad + \pi\left(\left[\rho^3 - \tfrac{1}{3}\rho^3\right] - \left[\rho^3\cos\phi - \tfrac{1}{3}\rho^3\cos^3\phi\right]\right)$$

$$= \quad '' \quad + \frac{2\pi}{3}\rho^3 - \pi\rho^3\cos\phi + \frac{\pi}{3}\rho^3 \underbrace{\cos^3\phi}_{=\cos\phi(1-\sin^2\phi}$$

$$= \cancel{\frac{\pi}{3}\rho^3 s^2 c} + \frac{2\pi}{3}\rho^3 - \pi\rho^3 c + \frac{\pi}{3}\rho^3 c - \cancel{\frac{\pi}{3}\rho^3 c s^2}$$

$$= \frac{2\pi}{3}\rho^3 - \frac{2\pi}{3}\rho^3 c = \frac{2\pi}{3}\rho^3(1-\cos\phi).$$

Step 2. Volume of a sphere is $\frac{4}{3}\pi\rho^3$. A spherical shell has volume $\frac{4}{3}\pi(\rho_2^3-\rho_1^3)$.

Let $\Delta\theta = \theta_2-\theta_1$. The volume of a "spherical slice" is $\frac{\Delta\theta}{2\pi}\cdot\frac{4}{3}\pi\rho^3 = \frac{2\Delta\theta}{3}\rho^3$.

Think ~~egg~~ orange slice

The volume of a slice of a spherical shell is $\frac{2\Delta\theta}{3}(\rho_2^3-\rho_1^3)$.

The fraction of a sphere between two cones $(\phi_1<\phi_2)$ is

$$\frac{\text{cone 2} - \text{cone 1}}{\text{Vol sphere}} =$$

$$\frac{\frac{2\pi}{3}\rho^3(1-\cos\phi_2) - \frac{2\pi}{3}\rho^3(1-\cos\phi_1)}{\frac{4\pi}{3}\rho^3} = \frac{1}{2}(\cos\phi_1-\cos\phi_2)$$

So, the volume of the spherical wedge is

$$\frac{\rho_2^3-\rho_1^3}{3}(\cos\phi_1-\cos\phi_2)(\theta_2-\theta_1).$$

$\square$

Theorem $\iiint f(\rho,\theta,\phi)\, dV = \iiint f(\rho,\theta,\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$.

Proof $\iiint f(\rho,\theta,\phi)\, dV = \lim\limits_{\substack{m\to\infty \\ n\to\infty \\ p\to\infty \\ \Delta V\to 0}} \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{p} f(\bar{\rho}_i, \bar{\theta}_j, \bar{\phi}_k)\, \Delta V_{ijk}$

where $\Delta V_{ijk} = \{(\rho,\theta,\phi) \mid \rho_i \le \rho \le \rho_{i+1},\ \theta_j \le \theta \le \theta_{j+1},$
$\phi_k \le \phi \le \phi_{k+1}\}$

and $(\bar{\rho}_i, \bar{\theta}_j, \bar{\phi}_k)$ is any point in ΔV_{ijk}.

The limit is not affected by how we choose $(\bar{\rho}_i, \bar{\theta}_j, \bar{\phi}_k)$.

Recall the MVT: For $f(x)$ differentiable and $a<b$, $\exists\ c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Let $f(\rho) = \frac{1}{3}\rho^3$. Then $\exists\ \bar{\rho}_i \in (\rho_i, \rho_{i+1})$ s.t.

$$\underbrace{f'(\bar{\rho}_i)}_{=\bar{\rho}_i^2} = \frac{\frac{1}{3}\rho_{i+1}^3 - \frac{1}{3}\rho_i^3}{\rho_{i+1}-\rho_i}$$

So $\frac{1}{3}\rho_{i+1}^3 - \frac{1}{3}\rho_i^3 = \bar{\rho}_i^2\, \Delta\rho_i$

Let $f(\phi) = \cos\phi$. $\exists\ \tilde{\phi}_j \in (\phi_j, \phi_{j+1})$ s.t.

$$-\sin\tilde{\phi}_j = \frac{\cos\phi_{j+1} - \cos\phi_j}{\phi_{j+1}-\phi_j}$$

Thus, $\cos \phi_j - \cos \phi_{j+1} = \sin \bar{\phi}_j \, \Delta \phi_i$.

~~[illegible]~~ Thus

$$\Delta V_{ijk} = \bar{\rho}_i^{\,2} \, \Delta \rho_i \, \sin \bar{\phi}_j \, \Delta \phi_j \, \Delta \theta_k$$

Thus $\lim \sum \sum \sum f(\quad) \, \bar{\rho}_i^{\,2} \sin \phi_j \, \Delta \rho_i \, \Delta \phi_j \, \Delta \theta_k$

$$= \int \int \int f(\rho, \theta, \phi) \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

for suitable bounds on the integrals.