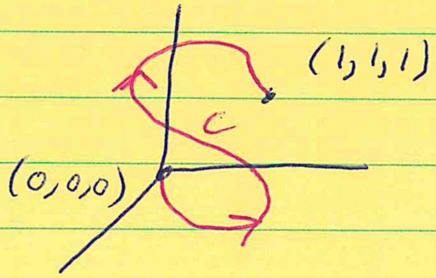
15 Extra Examples for Chapter 13

Ex 1 Let $F = \langle yz^2 + 3x^2 z, xz^2, 2xyz + x^3 \rangle$
Find the work done by F in pushing a particle along the path C shown.



Sol. (Outline) There is no hope of doing this problem unless F is conservative. Show that $\nabla \times F = 0$. Then show that $f = xyz^2 + x^3 z$ is a potential function for F, i.e. $F = \nabla f$. Thus, by the Fund. Thm of Line Integrals

$$\int_C F \cdot T \, ds = f(1,1,1) - f(0,0,0) = 2 .$$

Ex 2 Let $F = \langle x, -z, y \rangle$. Find $\int_C F \cdot T \, ds$ for the path in the last example or explain why this is an impossible task.

Sol. $\nabla \times F = \langle 2, 0, 0 \rangle$. Thus F is not conservative. Thus, we cannot solve this problem because we do not have a formula for the path taken or a means of finding one.

Ex 3 Let $F = \langle x, -z, y \rangle$. Let C be given by $r(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$. Find work done in push a particle along C.

Sol.

$$\int_C F \cdot T\, ds = \int_0^1 F \cdot \frac{dr}{dt}\, dt =$$

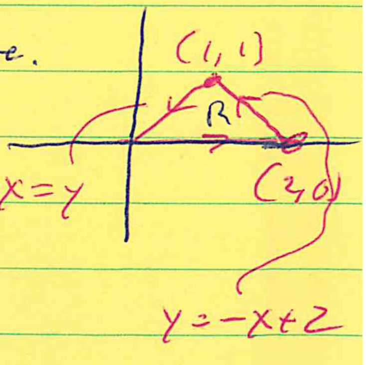
$$\int_0^1 \langle t, -t^3, t^2 \rangle \cdot \langle 1, 2t, 3t^2 \rangle\, dt =$$

$$\int_0^1 t - 2t^4 + 3t^4\, dt =$$

$$\int_0^1 t^4 + t\, dt = \frac{1}{5} + \frac{1}{2} = \frac{7}{10} = \underline{0.7}$$

Ex 4 Let $F = \langle xy, x^2+y^2 \rangle$. Let C be the triangle with vertices $(0,0)$, $(2,0)$, $(1,1)$ oriented ccw. Find $\int_C F \cdot dr$.

Sol. I'd use Green's Thm. Draw picture.



$$\partial_x (x^2+y^2) = 2x$$
$$\partial_y (xy) = x$$

$$\int_C F \cdot dr = \iint_R 2x - x\, dA =$$

$$\int_0^1 \int_y^{2-y} x\, dx\, dy = \ldots\ldots = 1.$$

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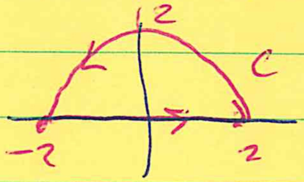
Ex (#18 13.4) A particle starts at the point $(-2,0)$, moves along the x-axis to $(2,0)$ and along the semi-circle $y=\sqrt{4-x^2}$ back to $(-2,0)$. If $F=\langle x, x^3+3xy^2\rangle$ is a force field, find the work done on the particle.

Sol. Draw!



I'd use Green's Thm.

$$\partial_x(x^3+3xy^2) = 3x^2+3y^2$$
$$\partial_y(x) = 0.$$

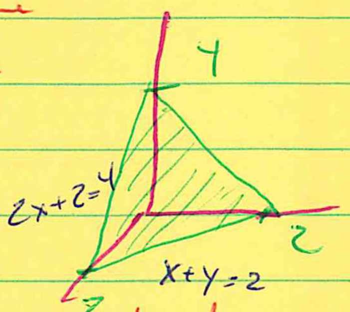
$$\int_c F\cdot dr = \iint 3x^2+3y^2\, dA$$

$$= \int_0^{\pi}\int_0^2 3\,\underbrace{r^2}_{}\, r\, dr\, d\theta \quad \text{(I'd use polar).}$$

$$= 3\pi \left.\frac{r^4}{4}\right|_0^2 = 3\pi\cdot 4 = \boxed{12\pi}.$$

Ex 6 (#10, 13.7) Compute $\iint_S xz\,dS$ where S is the portion of the plane $2x+2y+z=4$ in the 1st octant. ~~[illegible]~~.

Sol. There is no vector field in this problem. It is just a surface integral. Maybe xz is the charge density on a metal triangular plate and we want to find the total charge.



I'll do the problem two ways, dydx and dzdx.

dydx

$$z = 4-2x-2y.$$

$$dS = \sqrt{(z_x)^2+(z_y)^2+1}\;dy\,dx$$

$$= \sqrt{4+4+1}\;dy\,dx$$

$$= 3\,dy\,dx$$

$$\iint xz\,dS =$$

$$3\int_0^2\int_0^{2-x} x(4-2x-2y)\,dy\,dx$$

$$= \cdots = 4.$$

dzdx

$$y = \frac{4-z-2x}{x}$$

$$dS = \sqrt{(y_x)^2+(y_z)^2+1}\;dz\,dx$$

$$= \sqrt{1+\tfrac{1}{4}+1} = \frac{3}{2}$$

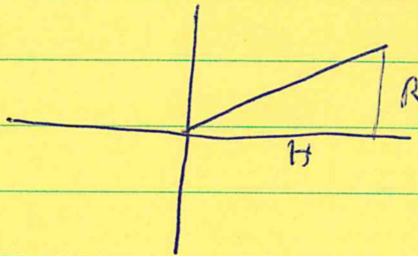
$$\iint xz\,dS =$$

$$\frac{3}{2}\int_0^2\int_0^{4-2x} xz\;dz\,dx$$

$$= \cdots = 4.$$

Ex 7 Find a formula for the surface area of a cone of height H and base ~~R~~ radias R. (without the base)

Sol.



$y = \frac{R}{H}x.$

Rotate this line segment shown about the axis. Then using formula $\boxed{10}$ on page 804 that was derived in the lecture notes, see also page 395 formula $\boxed{4}$, we have,

$$S.A. = \int_0^H 2\pi\left(\frac{R}{H}x\right)\sqrt{1+\frac{R^2}{H^2}}\,dx$$

$$= \frac{2\pi R}{H}\frac{H^2}{2}\sqrt{1+\frac{R^2}{H^2}} = \pi RH\sqrt{1+\frac{R^2}{H^2}}$$

$$= \pi R\sqrt{H^2+R^2}.$$

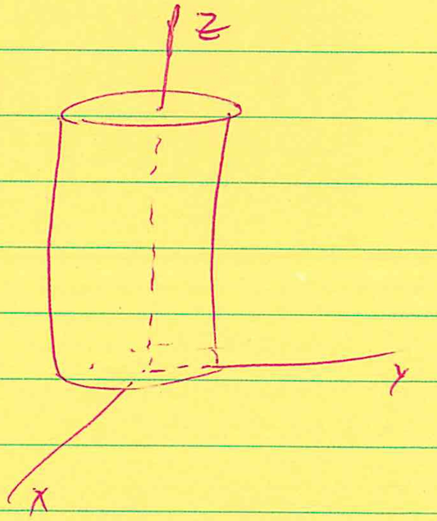
Ex 8 Consider a metal ~~cylinder~~ tube of radius 2 and height 10. There is no top or bottom. ~~The cha~~ It is positioned so that the z-axis is its central axis, and the base rests on the xy-plane. The charge density is $f(x,y) = x^2 z$. Find the total charge, Q.

Sol. (outline) $Q = \iint x^2 z \, dS.$

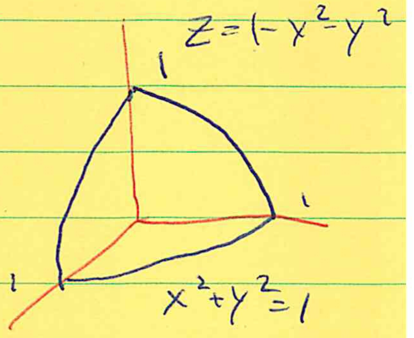
Parameterize the tube: $r(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z \rangle$.

Check that $|r_\theta \times r_z| = 2$.

Then $Q = \int_0^{10} \int_0^{2\pi} (2\cos\theta)^2 z \cdot 2 \, d\theta \, dz = 400\pi$.



Ex 9 Let $F = \langle x, xy, xz \rangle$. Let S be given by $z = 1 - x^2 - y^2$, $z \geq 0$, oriented upward. Find the flux of F over S.



Sol $\text{Flux} = \iint F \cdot N \, dS$

$$N = \langle \overset{2x}{-z_x}, \overset{2y}{-z_y}, 1 \rangle \Big/ \sqrt{(z_x)^2 + (z_y)^2 + 1}$$

$$dS = \sqrt{(z_x)^2 + (z_y)^2 + 1} \; dA \longrightarrow \text{I'll use polar in a bit.}$$

$$F \cdot N = \left(2x^2 + 2xy^2 + x(1 - x^2 - y^2) \cdot 1\right) \Big/ \sqrt{}$$

$$= \left(-x^3 + xy^2 + 2x^2 + x\right) \Big/ \sqrt{}$$

$$\iint F \cdot N \, dS = \iint -x^3 + xy^2 + 2x^2 + x \; dA$$

$$= \int_0^{2\pi} \int_0^1 \left[-r^3 \underset{=0}{\cos^3(\theta)} + r^3 \underset{=0}{\cos\theta \sin^2\theta} + 2r^2 \underset{=\pi}{\cos^2\theta} + r \underset{=0}{\cos\theta} \right] r \, dr \, d\theta$$

$$2\pi \int_0^1 r^3 \, dr = \frac{\pi}{2}$$

Ex 10 (#28 13.7) Let $F = \langle xy, 4x^2, yz \rangle$. Let S be given by $z = xe^y$, $0 \le x \le 1$, $0 \le y \le 1$. Find the flux of F up through S.

Sol $\iint_S F \cdot N \, dS$.

$$N = \frac{\langle -z_x, -z_y, 1 \rangle}{\sqrt{(z_x)^2 + (z_y)^2 + 1^2}}$$

$$dS = \sqrt{(z_x)^2 + (z_y)^2 + 1^2} \, dx dy$$

$$z_x = e^y \quad z_y = xe^y$$

$$F \cdot N = \langle xy, 4x^2, y(xe^y) \rangle \cdot \langle -e^y, -xe^y, 1 \rangle / \sqrt{}$$

$$= \left(-\underline{xy}e^y - 4x^3e^y + \underline{yx}e^y\right) / \sqrt{} = \frac{-4x^3e^y}{\sqrt{}}$$

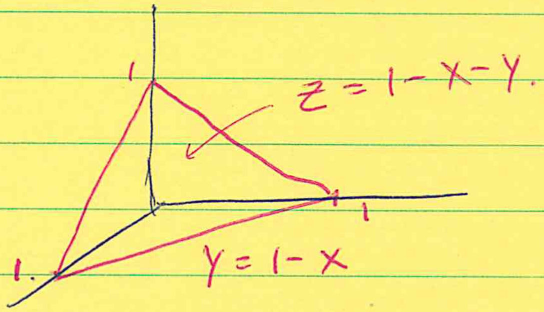
$$\iint F \cdot N \, dS = \int_0^1 \int_0^1 -4x^3 e^y \, dx dy$$

$$= e - 1.$$

Ex 11 Let $F = \langle xy, yz, xz \rangle$. Find the flux of F out of the region R determined by

$$0 \le x \le 1,$$
$$0 \le y \le 1-x,$$
$$0 \le z \le 1-x-y.$$

Sol. Draw region.



You could find the flux by integrating over the four sides, or you could use the divergence thm! I'd use the divergence thm.

$$\operatorname{div} F = y + z + x.$$

$$\text{Flux} = \iint_{\partial R} F \cdot N \, dS = \iiint_{R} \operatorname{div} F \, dV$$

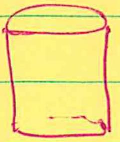
$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x + y + z \, dz \, dy \, dx = \frac{1}{8}.$$

Ex 12 Let $F = \langle x, 2y^2, 3z^2 \rangle$.

Let R be the region $0 \leq x^2+y^2 \leq 9 \quad 0 \leq z \leq 1$.

Find the flux of F of R.

Sol. We can do the surface integrals over the top, bottom and side. Or we can use the divergence thm! I'd use the later.



$$\text{Flux} = \iint_{\partial R} F \cdot N \, dS = \iiint_{R} \operatorname{div} F \, dV.$$

$$\operatorname{div} F = 1 + 4y + 6z$$

$$\text{Flux} = \int_0^1 \int_0^{2\pi} \int_0^3 (1 + 4y + 6z) \, r \, dr \, d\theta \, dz$$

$$= \int_0^1 \int_0^{2\pi} \int_0^3 (1 + 4r\sin\theta + 6z) r \, dr \, d\theta \, dz$$

$$r + 4r^2 \sin\theta + 6zr$$

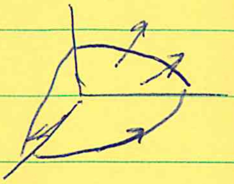
$$= \int_0^1 \int_0^{2\pi} \frac{9}{2} + 4 \cdot 9 \sin\theta + 6z \frac{9}{2} \, d\theta \, dz$$

$$= \int_0^1 \frac{9}{2} \cdot 2\pi + 0 + 27z \cdot 2\pi \, dz$$

$$= 9\pi + 27\pi = 36\pi.$$

Ex 13 Let S be the top half of the unit sphere $x^2+y^2+z^2=1$ oriented upward. Let $F = \langle z, x, xyz \rangle$.
Find $\iint_S (\nabla \times F)\cdot N \, dS$.

Sol. I By stokes' thm this is $= \oint_c F \cdot T \, ds$ where C is $x^2+y^2=1$ $(z=0)$, ccw.



Let $r(t) = \langle \cos t, \sin t, 0 \rangle$.
Then $r'(t) = \langle -\sin t, \cos t, 0 \rangle$
$F \cdot \frac{dr}{dt} = \langle 0, \cos t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle = \cos^2 t$

$$\oint F \cdot \frac{dr}{dt} \, dt = \int_0^{2\pi} \cos^2 t \, dt = \boxed{\pi}.$$

Sol II By direct calculation.

$\nabla \times F = \langle xz, 1-yz, 1 \rangle$.
$N = \langle x, y, z \rangle$
$dS = \sin\phi \, d\phi \, d\theta$

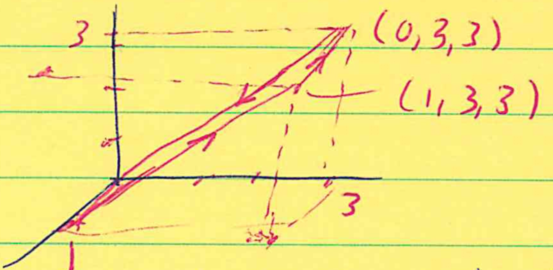
$$\iint \nabla \times F \cdot N \, dS = \int_0^{2\pi}\int_0^{\pi/2} (x^2 z + y - y^2 z + z) \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} (\cos^2\theta - \sin^2\theta)\sin^3\phi\cos\phi + \sin\theta\sin^2\phi + \cos\phi\sin\phi \quad d\phi \, d\theta$$

$$= \int_0^{2\pi} \cos(2\theta)\cdot\frac{1}{4} + \sin\theta\cdot\frac{\pi}{4} + \frac{1}{2} \, d\theta = 0 + 0 + \frac{2\pi}{2} = \boxed{\pi}.$$

Ex 14 Let $F = \langle x^2, 4xy^3, xy^2 \rangle$. Let C be the retangle with vetrices $(0,0,0)$, $(0,3,3)$, $(1,3,3)$, $(1,0,0)$. Find the work done by F in pushing a particle once around C, ccw when viewed from above.

Sol. Draw picture.



We want to find $\oint_C F \cdot T\, ds$. We could break this up into four line integrals. Instead we'll use Stokes' Thm.

$$\oint_C F \cdot T\, ds = \iint_S (\nabla \times F) \cdot N\, dS.$$

By inspection we see the equation of the plane containing our retangle is $y = z$, or $z = y$. We will do a dxdy integral.

$$N = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle \Big/ \sqrt{} = \langle 0, -1, 1 \rangle / \sqrt{2}.$$

$$dS = \sqrt{2}\, dx\,dy.$$

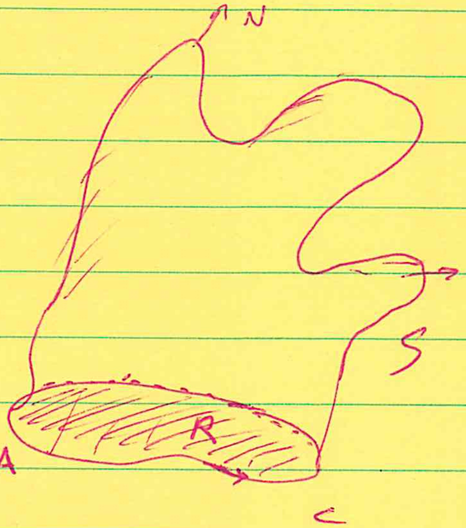
$$\nabla \times F = \langle 2xy, -y^2, 4y^3 \rangle \qquad \text{(check this)}$$

$$\iint \nabla \times F \cdot N\, dS = \int_0^3 \int_0^1 \langle 2xy, -y^2, 4y^3 \rangle \cdot \langle 0, -1, 1 \rangle\, dx\,dy$$

$$= \int_0^3 \int_0^1 y^2 + 4y^3\, dx\,dy = \cdots = \underline{90.}$$

Ex 15 Let S be the surface shown. The surface area is 13. The boundary of S is the curve C that enclos[illegible]es a region R in the xy-plane of area 8. Let $F = \langle [illegible] \cdot y + e^{z}, 5x + z^{3}, xyz \rangle$. Find the flux of $\nabla \times F$ over S.

Sol. $\nabla \times F = \langle xz - 3z^{2}, e^{z} - yz, 4 \rangle$.



Flux $= \iint \nabla \times F \cdot N \, dS$.

But, we cannot find N or dS!

But, we can still do this problem.

$$\iint_S \nabla \times F \cdot N \, dS = \oint_C F \cdot T \, ds = \iint_R \nabla \times F \cdot \langle 0,0,1 \rangle \, dA$$

$$= 4 \iint_R dA = 4 \times 8 = \underline{24.}$$