10.3 The dot product

Def Let $v = \langle v_1, v_2, v_3 \rangle$, $w = \langle w_1, w_2, w_3 \rangle$.

Then $v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3 \in \mathbb{R}$.

Ex $(2i + 2j - k) \cdot (3i + j + k) = 6 + 2 - 1 = 7$.

$\langle 3, 4, 2 \rangle \cdot \langle 1, 1, 2 \rangle = 3 + 4 + 4 = 11$.

$\langle 2, 1 \rangle \cdot \langle 1, 3 \rangle = 2 + 3 = 5$.

Fact $|v| = \sqrt{v \cdot v}$

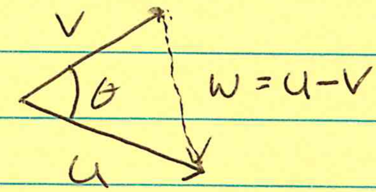
Properties Let u, v and w be vectors and $r \in \mathbb{R}$.

1. $v \cdot v = |v|^2$
2. $v \cdot w = w \cdot v$
3. $u \cdot (v + w) = u \cdot v + u \cdot w$
4. $(ru) \cdot v = r(u \cdot v) = u \cdot (rv)$
5. $\mathbf{0} \cdot v = 0$ the number

You should be able to prove these!

Angles

Thm $\quad v \cdot u = |v||u| \cos\theta$



Pf $\quad$ Recall the Law of Cosines:

$$|u-v|^2 = |v|^2 + |u|^2 - 2|v||u|\cos\theta$$

When $\theta = \frac{\pi}{2}$ this becomes Pythagorean Thm

But we also know from the properties of vectors that

$$\begin{aligned} |u-v|^2 &= (u-v)\cdot(u-v) = (u-v)\cdot u + (u-v)\cdot(-v) \\ &= u\cdot u - v\cdot u - u\cdot v + v\cdot v \\ &= |u|^2 - 2(v\cdot u) + |v|^2. \end{aligned}$$

Thus,

$$\cancel{|u|^2} - 2(v\cdot u) + \cancel{|v|^2} = \cancel{|v|^2} + \cancel{|u|^2} - 2|v||u|\cos\theta,$$

implies $\quad v\cdot u = |v||u|\cos\theta. \qquad \square$

This is also written as

$$\cos\theta = \frac{v\cdot u}{|v||u|}.$$

Ex Find the angle between $\langle 1,2,1\rangle$ and $\langle 0,1,1\rangle$.

Sol
$$\cos\theta = \frac{\langle 1,2,1\rangle \cdot \langle 0,1,1\rangle}{|\langle 1,2,1\rangle|\,|\langle 0,1,1\rangle|} = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{\sqrt{3}}{2}.$$

Thus, $\theta = \frac{\pi}{6}$ rad. or $30°$.

Notation $V \perp U$ means V is perpendictular to U.

Fact If $V \perp U$, then $\theta = \frac{\pi}{2}$ and thus $V \cdot U = 0$.

We define $V \perp \mathbf{0}$ for any V. Then

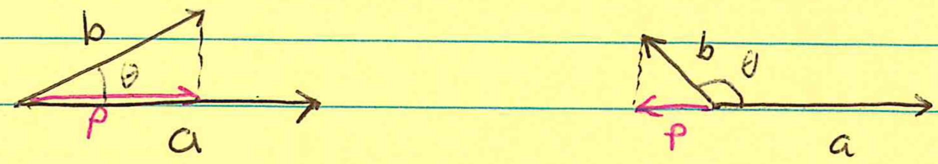
$$V \perp U \iff V \cdot U = 0.$$

Ex Find a real number x such that $\langle 1,2,1\rangle$ and $\langle x,1,1\rangle$ are $\perp$.

Sol
$$\langle 1,2,1\rangle \cdot \langle x,1,1\rangle = x+3.$$

This is zero only when $x=-3$.

Projections and Components



$\text{proj}_a b$ = projection of b onto a.

$\text{comp}_a b$ = magnitude and "sign" of $\text{proj}_a b$.
(sign = + directions are the same,
sign = − directions are opposite)

Thm (i) $\text{proj}_a b = \dfrac{a \cdot b}{|a|^2} a$ (ii) $\text{comp}_a b = \dfrac{a \cdot b}{|a|}$

pf (ii) follows from (i):

$$\frac{a \cdot b}{|a|^2} a = \frac{a \cdot b}{|a|} \frac{a}{|a|}$$ ⇐ unit vector.

If $\theta < 90^\circ$, $a \cdot b > 0$. If $\theta > 90^\circ$, $a \cdot b < 0$.

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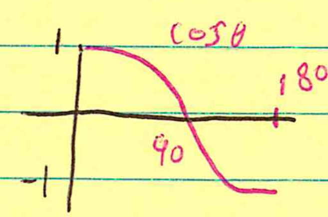
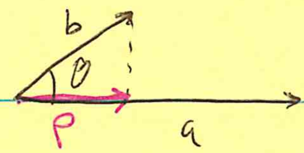


① 

$\pm |p| = |b| \cos\theta$

Thus, $p = |b| \cos\theta \dfrac{a}{|a|} = \cancel{|b|} \dfrac{a \cdot b}{|a|\cancel{|b|}} \dfrac{a}{|a|}$

$$= \frac{a \cdot b}{|a|^2} a \qquad \square$$

<u>Ex</u> Project $\langle 1, 2, 1 \rangle$ onto $\langle 1, 1, 1 \rangle$.

<u>Sol</u> $\langle 1, 2, 1 \rangle \cdot \langle 1, 1, 1 \rangle = 4$

$|\langle 1, 1, 1 \rangle| = \sqrt{3}$.

Thus, $\mathrm{proj}_{\langle 1,1,1 \rangle} \langle 1, 2, 1 \rangle = \frac{4}{3} \langle 1, 1, 1 \rangle = \langle \frac{4}{3}, \frac{4}{3}, \frac{4}{3} \rangle$.