10.4 The Cross Product

Def Let $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$. Then

$$u \times v = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - v_2 v_1 \rangle.$$

Shortcut $u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} =$

$$(u_2 v_3 - u_3 v_2) i - (u_1 v_3 - u_3 v_1) j + (u_1 v_2 - v_2 v_1) k.$$

Ex Pause the video and check the following

$\langle 1, 2, 1 \rangle \times \langle 3, 1, -1 \rangle = \langle -3, 4, -5 \rangle$.

$\langle 2, 1, 3 \rangle \times \langle 4, 7, 1 \rangle = \langle -20, 10, 10 \rangle$.

$i \times j = k$, $j \times i = -k$,

For any vector v in $\mathbb{R}^3$ show that

$$v \times v = \mathbf{0}.$$

the zero vector!

Properties Let a, b, c and v be vectors in $\mathbb{R}^3$. Let $r \in \mathbb{R}$.

1. $v \times 0 = 0$, $v \times v = 0$
2. $a \times b = -b \times a$
3. $(ra) \times b = r(a \times b) = a \times (rb)$
4. $a \times (b + c) = a \times b + a \times c$
5. $(a + b) \times c = a \times c + b \times c$
6. $a \cdot (b \times c) = (a \times b) \cdot c$
7. $a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$

Note The cross product is not associative!

Ex $(i \times i) \times j = 0 \times j = 0$

$i \times (i \times j) = i \times k = -j$ (check this!)

Proof of Property 4

$$a \times (b + c) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix} =$$

$$\langle a_2(b_3 + c_3) - a_3(b_2 + c_2),\ a_3(b_1 + c_1) - a_1(b_3 + c_3),\ a_1(b_2 + c_2) - a_2(b_1 + c_1) \rangle$$

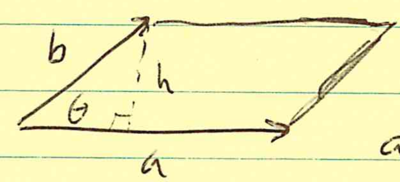
$$
\begin{aligned}
&= \langle a_2 b_3 + a_2 c_3 - a_3 b_2 - a_3 c_2 ,\ a_3 b_1 + a_3 c_1 - a_1 b_3 - a_1 c_3 , \\
&\qquad\qquad a_1 b_2 + a_1 c_2 - a_2 b_1 - a_2 c_1 \rangle \\
&= \langle a_2 b_3 - a_3 b_2 + a_2 c_3 - a_3 c_2 ,\ a_3 b_1 - a_1 b_3 + a_3 c_1 - a_1 c_3 , \\
&\qquad\qquad a_1 b_2 - a_2 b_1 + a_1 c_2 - a_2 c_1 \rangle \\
&= \langle a_2 b_3 - a_3 b_2 ,\ a_3 b_1 - a_1 b_3 ,\ a_1 b_2 - a_2 b_1 \rangle \\
&\qquad + \langle a_2 c_3 - a_3 c_2 ,\ a_3 c_1 - a_1 c_3 ,\ a_1 c_2 - a_2 c_1 \rangle \\
&= a \times b + a \times c .
\end{aligned}
$$

Geometric Properties

1. $a \times b$ is $\perp$ a and $\perp$ b.

Pf $(a \times b) \cdot b = a \cdot (b \times b) = a \cdot 0 = 0.$
$(a \times b) \cdot a = a \cdot (a \times b) = (a \times a) \cdot b = 0 \cdot b = 0.$

2. $|a \times b| = |a||b| \sin\theta =$ area of parallelogram

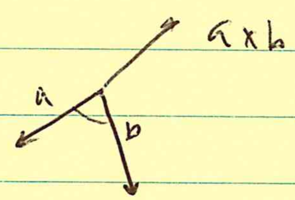


$h = |b| \sin\theta$
area $= |a||b| \sin\theta$

Pf: Proof is hard. See page 560.

3. $a \times b = 0 \iff a \parallel b.$

RHR:



Phys. Def: $a \times b$ is the unique vector with mag. the area of the parallelogram determined by a and b, that is $\perp$ to both a and b and follows the RHR.

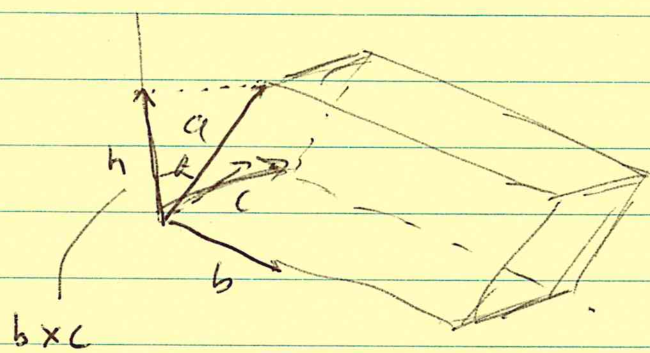
Def Given vectors a, b, c the <u>scalar triple prod</u> is

$$a \cdot (b \times c).$$

Fact $a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

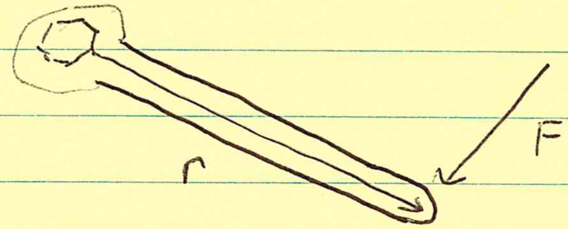
Thm The volume of the parallelepiped determined by a, b, c is $|a \cdot (b \times c)|$.

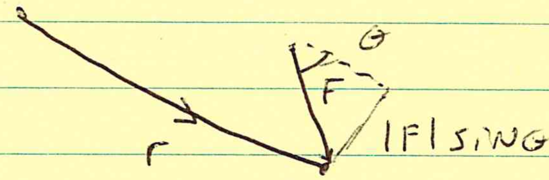
Rough Pf



$$|a \cdot (b \times c)| = |a|\,|b \times c|\cos\theta$$

Torque





mag of torque $|\tau| = |r||F| \sin\theta$.

It direction is papallel to the axis of rotation, which is $\perp$ to r and F. such that

$$\tau = r \times F$$