10.5 Lines and Planes

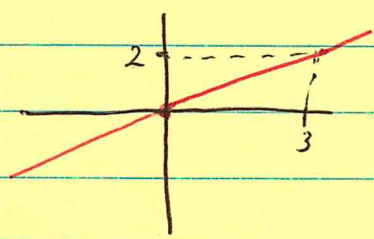
$\mathbb{R}^2$ We review lines in $\mathbb{R}^2$. Every line L in $\mathbb{R}^2$ can be expressed in the form

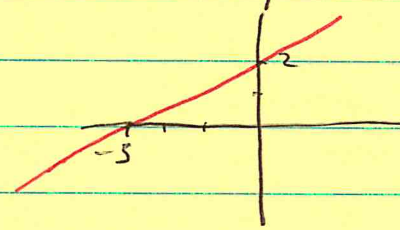
$$Ax + By = C.$$

Call this the ABC form. It is not unique since multiplication by any nonzero constant does not change L.

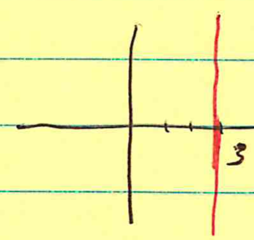
Recall the slope-intercept form, $y = ax + b$, is unique but cannot by applied to vertical lines.

Ex $-2x + 3y = 0$ $\qquad$ $-2x + 3y = 6$





$x = 3$ $(A = 1, B = 0, C = 3)$ is vertical



$A = 0, B = 0, C = 1$ is not a line.
It is the empty set.

$A = 0, B = 0, C = 0$ is not a line.
It is all of $\mathbb{R}^2$.

We give a geometric interpetation for the ABC form. First, suppose $C=0$, and at least one of A or B is nonzero.

$$Ax+By=0.$$

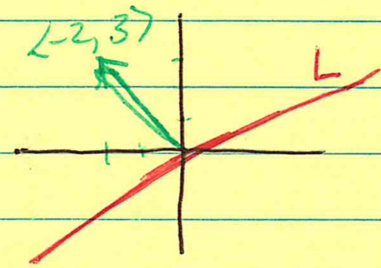
The line L will go through the origin. Let

$$n=\langle A,B\rangle \text{ and } p=\langle x,y\rangle.$$

Then we have $n\cdot p=0$.

Thus, L is the set of points, $(x,y)\in\mathbb{R}^2$, such that $\langle x,y\rangle$ is perpendicular to $\langle A,B\rangle$.

<u>Ex</u> $-2x+3y=0$



Now suppose $C\neq 0$. Pick some point (x_0,y_0) on L. Then

$$Ax+By=C=Ax_0+By_0$$

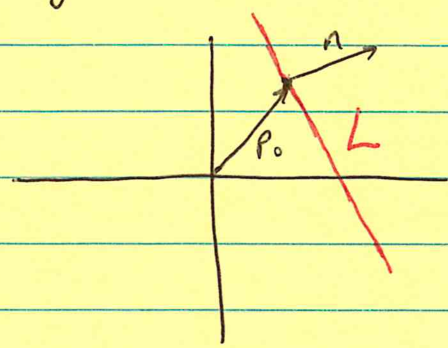
$$\Rightarrow \quad A(x-x_0)+B(y-y_0)=0$$

$$\langle A,B\rangle\cdot\langle x-x_0,y-y_0\rangle=0$$

$$\langle A,B\rangle\cdot\left(\langle x,y\rangle-\langle x_0,y_0\rangle\right)=0$$

$$n\cdot(p-p_0)=0.$$

The vector $p - p_0$ can be thought of as lying in L with it base at (x_0, y_0) and its head at (x, y). Thus the line L is the unique line perpendicular to $n = \langle A, B \rangle$ that passes through (x_0, y_0).



<u>Parametric Form</u> Recall that any line L in $\mathbb{R}^2$ can be expressed as parametric equations

$$x(t) = at + b,$$
$$y(t) = ct + d.$$

These can be written in vector form as

$$\langle x, y \rangle = \langle a, c \rangle t + \langle b, d \rangle.$$

Let $p(t) = \langle x(t), y(t) \rangle$, and regard t as time.

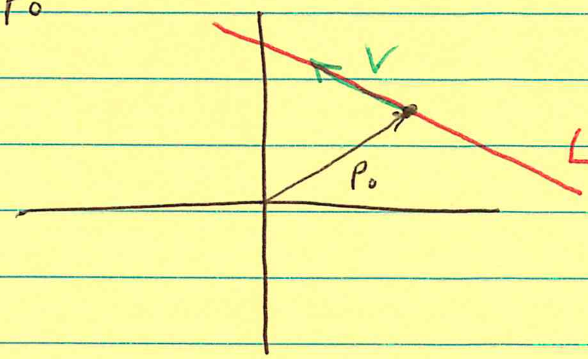
Then $p(0) = \langle b, d \rangle$ is the initial position and

$\frac{dp}{dt} = \langle x'(t), y'(t) \rangle = \langle a, c \rangle$ is the velocity vector.

Now we can rewrite our parametric equations as

$$p(t) = vt + p_0$$

(where $p_0 = p(0)$).



Pause the video and do these two problems.

Prob 1 Let $x(t) = 3t - 2$, $y(t) = -t + 7$. Find an ABC form for this line.

Prob 2 Let $4x - 7y = 2$. Find parametric equations for this line.

Each has many correct answers.

Lines in $\mathbb{R}^3$

In $\mathbb{R}^3$ a line L can be expressed by parametric equations

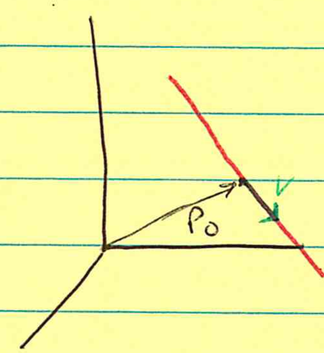
$$x(t) = at + b$$
$$y(t) = ct + d$$
$$z(t) = et + f.$$

In vector form: $\langle x(t), y(t), z(t)\rangle = \langle a, c, e\rangle t + \langle b, d, f\rangle$.
Let $v = \langle a, c, e\rangle$, $p(t) = \langle x(t), y(t), z(t)\rangle$ and $p_0 = p(0)$
$= \langle b, d, f\rangle$.

Then

$$p(t) = vt + p_0$$

and $p'(t) = v$.



Ex $p(t) = \langle 1, 2, 3\rangle t + \langle 1, 1, 1\rangle$, $p(t) = \langle 2, 4, 6\rangle t + \langle 1, 1, 1\rangle$
and
$$p(t) = \langle 1, 2, 3\rangle t + \langle 2, 3, 4\rangle$$

all give the same line in $\mathbb{R}^3$.

Planes in $\mathbb{R}^3$.

Consider the solution set to $Ax+By+Cz=D$ in $\mathbb{R}^3$.

If $A=B=C=D=0$, the solution set is all of $\mathbb{R}^3$.
If $A=B=C=0$ and $D\neq 0$, the solution set is empty.

Otherwise the solution set is a plane.

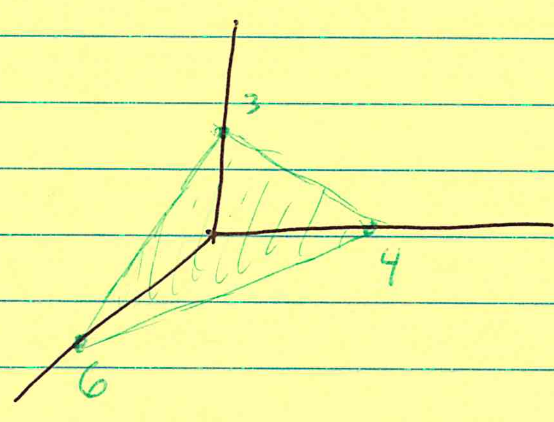
Ex 1 If $A=B=D=0$ and $C\neq 0$, then the equation is

$$Cz=0$$

or $$z=0$$

which gives the xy-plane as x and y have no restrictions.

Ex 2 Graph $2x+3y+4z=12$.



Next we give a geometric interpretation of an equation for a plane P given in <u>ABCD form</u>. First, suppose $D=0$. Then

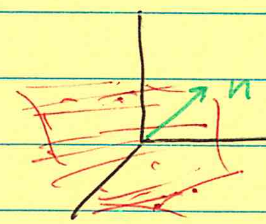
$$Ax+By+Cz=0.$$

$$\Rightarrow \qquad \langle A,B,C\rangle\cdot\langle x,y,z\rangle=0.$$

Let $n=\langle A,B,C\rangle$ and $p=\langle x,y,z\rangle$.
Now

$$n\cdot p=0.$$

Thus our plane P passes through the origin and is $\perp$ to n.



For the case where $D\neq 0$ let $p_0=\langle x_0,y_0,z_0\rangle$ be any point on the plane. Then as before

$$n\cdot(p-p_0)=0$$

but now $n=\langle A,B,C\rangle$, $p=\langle x,y,z\rangle$, $p_0=\langle x_0,y_0,z_0\rangle$.

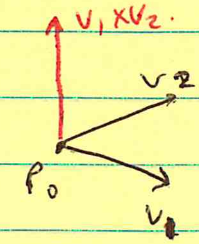
The plane passes through (x_0,y_0,z_0) and is $\perp$ to $\langle A,B,C\rangle$.

Ex

Find an equation for the plane passing through the points $(1,1,1)$, $(1,2,3)$ and $(2,-1,0)$.

Sol 1 We want an equation of the form $n \cdot (P - P_0) = 0$. Let $P_0 = \langle 1,1,1 \rangle$. Let

$$V_1 = \langle 1,2,3 \rangle - \langle 1,1,1 \rangle = \langle 0,1,2 \rangle$$
$$V_2 = \langle 2,-1,0 \rangle - \langle 1,1,1 \rangle = \langle 1,-2,-1 \rangle.$$



Then let

$$n = v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= \langle 3, 2, -1 \rangle.$$

Thus, we have

$$\langle 3,2,-1 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 0$$

$$3x - 3 + 2y - 2 - z + 1 = 0$$

$$3x + 2y - z = 4.$$

Sol 2 We have three conditions and this give us three equations in four unknows.

$$\begin{aligned} A + B + C &= D & (1,1,1) \\ A + 2B + 3C &= D & (1,2,3) \\ 2A - B &= D & (2,-1,0) \end{aligned}$$

$$\begin{aligned} A + B + C &= D & \\ B + 2C &= 0 & ((2)-(1)) \\ -3B - 2C &= -D & ((3)-2(1)) \end{aligned}$$

$$\begin{aligned} A + B + C &= D & \\ B + 2C &= 0 & \\ 4C &= -D & ((3)+3(2)) \end{aligned}$$

Thus, $C = -D/4$. Hence, $B = -2C = D/2$ and

$$A = D - B - C = D - D/2 + D/4 = 3D/4.$$

Any nonzero value for D will work. Let $D = 4$.

$$3x + 2y - z = 4.$$

This method is harder, but generalizes to higher dimensions. See MATH 221.

Ex Let P_1 be the plane given by $2x+3y-z=4$.
Let P_2 be the plane given by $x+y+z=1$.
Find parametric equations for the line $L = P_1 \cap P_2$.

Sol
$$\left.\begin{aligned} 2x+3y-z&=4 \\ x+y+z&=1 \end{aligned}\right\} \Rightarrow 3x+4y=5$$

Let $x=t$. Then $y = \frac{5-3x}{4} = \frac{5}{4} - \frac{3}{4}t$.

And $z = 1-x-y = 1-t-\frac{5}{4}+\frac{3}{4}t = -\frac{1}{4}-\frac{1}{4}t$

$$\begin{aligned} x &= t \\ y &= -\tfrac{3}{4}t + \tfrac{5}{4} \\ z &= -\tfrac{1}{4}t - \tfrac{1}{4} \end{aligned}$$

In vector form $\langle x\ y\ z\rangle = \langle 1, -\frac{3}{4}, -\frac{1}{4}\rangle t + \langle 0, \frac{5}{4}, -\frac{1}{4}\rangle$.

Ex Consider the plane $x + 2y - z = 2$ and the line $\langle x, y, z \rangle = \langle 1, 1, 1 \rangle t + \langle 1, 2, 4 \rangle$. Find the point where they meet.

Sol $x = t+1$, $y = t+2$, $z = t+4$. Thus,

$$(t+1) + 2(t+2) - (t+4) = 2.$$

$$2t + 1 = 2$$

$$t = \frac{1}{2}.$$

Thus, the intersection pt is $\left(\frac{3}{2}, \frac{5}{2}, \frac{9}{2}\right)$.