10.7 Curves in $\mathbb{R}^3$

Ex First in $\mathbb{R}^2$ (a) $r(t) = \langle t, t^2 \rangle$ (b) $r(t) = \langle 4t, 16t^2 \rangle$, $r(t) = \langle e^t, e^{2t} \rangle$, $r(t) = \langle \sin t, \sin^2 t \rangle$.

Ex $\langle t, t^2, 0 \rangle$, $\langle t, t^2, 5 \rangle$, $\langle t, t^2, t \rangle$

Ex $\langle \cos t, \sin t, 0 \rangle$, $\langle \cos t, \sin t, t \rangle$ helix

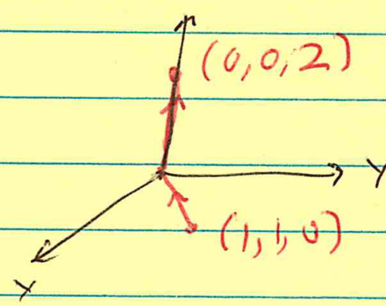
Ex Find parametric eq. for the line segment that starts at $(1,1,2)$ and goes to $(0,4,3)$ as t goes from 0 to 1.

$x = 1 - t$

$y = 1 + 3t$ $\qquad r(t) = \langle -1, 3, 1 \rangle t + \langle 1, 1, 2 \rangle$.

$z = 2 + t$

Ex Find a parametric eq. for →



For $0 \le t \le 1$ use

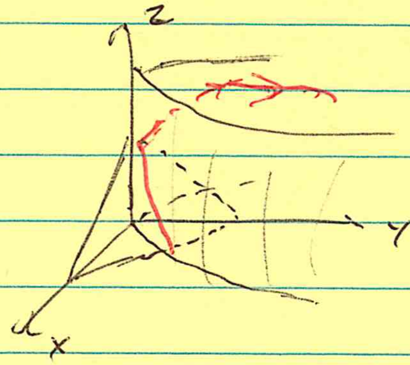
$r_1(t) = \langle 1-t, 1-t, 0 \rangle$;

For $1 \le t \le 3$ use

$r_2(t) = \langle 0, 0, t-1 \rangle$.

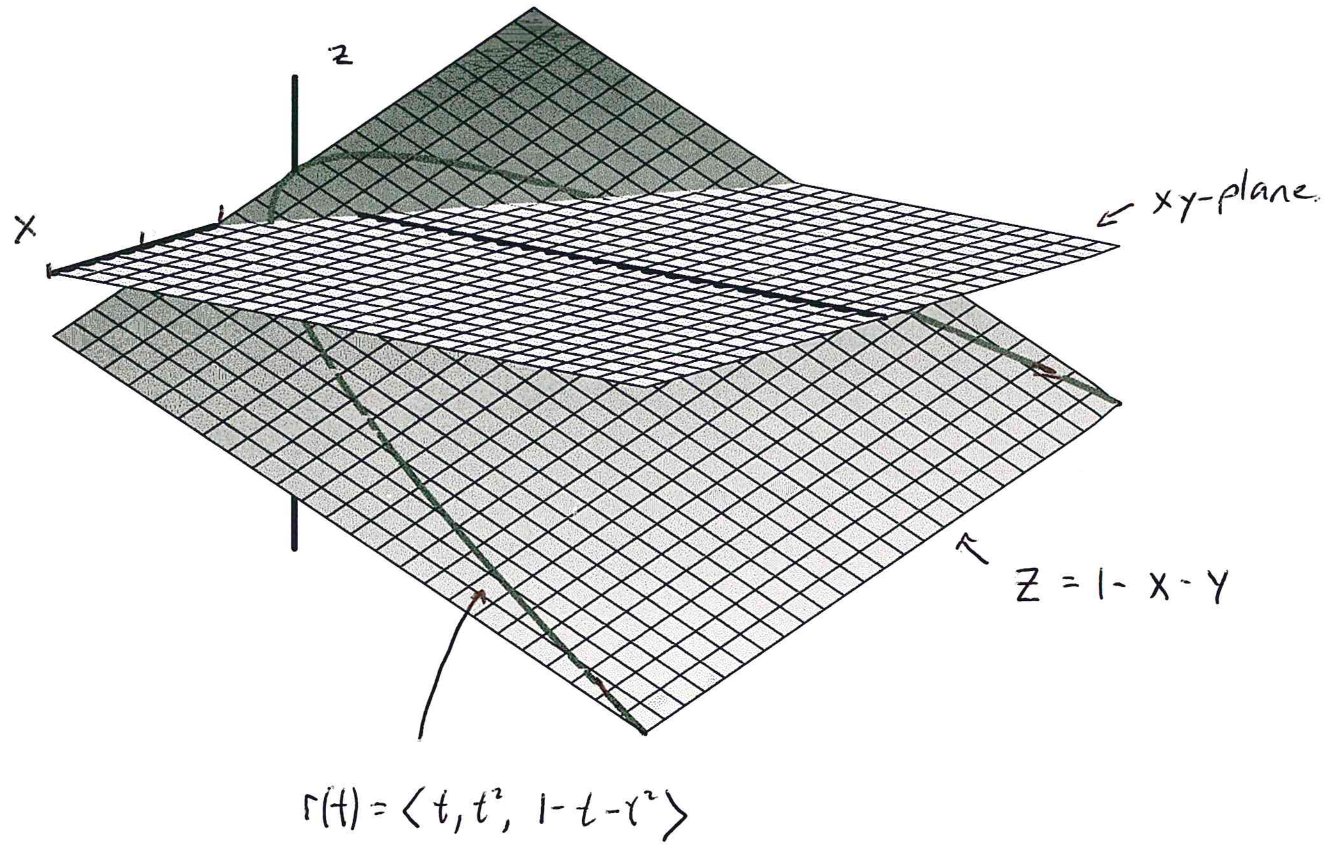
Ex Find a parabolic eq. for the curve formed by the intersection of $y=x^2$ and $x+y+z=1$.

Sol Let $x=t$.
Then $y=t^2$ and
$z = 1-x-y = 1-t-t^2$



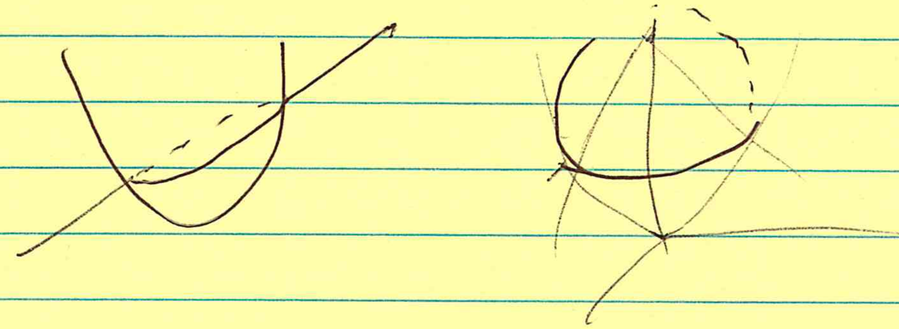
Question What is the maximum height above the xy-plane of this curve?

Sol $\frac{dz}{dt} = -1-2t \overset{set}{=} 0$. Then $t = -\frac{1}{2}$.
Thus $z = 1-(-\frac{1}{2})-(-\frac{1}{2})^2 = 1+\frac{1}{2}-\frac{1}{4} = 1\frac{1}{4} = 1.25$.


z
x
xy-plane
Z = 1 - X - Y
r(t) = ⟨t, t², 1-t-t²⟩

Ex Find a parametric eq for the curve formed by the intersection of $z = x^2+y^2$ and $x+y+z=4$.

Sol Notice the curve is a closed loop.



I will project this loop into the xy-plane by eliminating z.

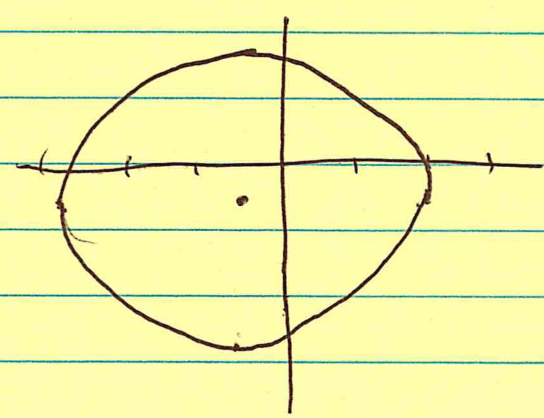
$$z = 4-x-y \quad \text{and} \quad z = x^2+y^2.$$

Thus $$x^2+y^2 = 4-x-y.$$

$$x^2+x+\tfrac{1}{4}+y^2+y+\tfrac{1}{4} = 4+\tfrac{1}{4}+\tfrac{1}{4}$$

$$(x+\tfrac{1}{2})^2+(y+\tfrac{1}{2})^2 = 4\tfrac{1}{2} = \tfrac{9}{2}$$

$$R = 3/\sqrt{2} \approx 2.12$$



We will parameterize this circle.

$$\left.\begin{aligned} x &= \tfrac{3}{\sqrt{2}}\cos t \\ y &= 3/\sqrt{2}\,\sin t \end{aligned}\right\}$$ gives circle radius $\frac{3}{\sqrt{2}}$ with center $(0,0)$.

Now, we "push it" to have center $(-\frac{1}{2}, -\frac{1}{2})$

$$x = \frac{3}{\sqrt{2}}\cos t - \frac{1}{2}$$

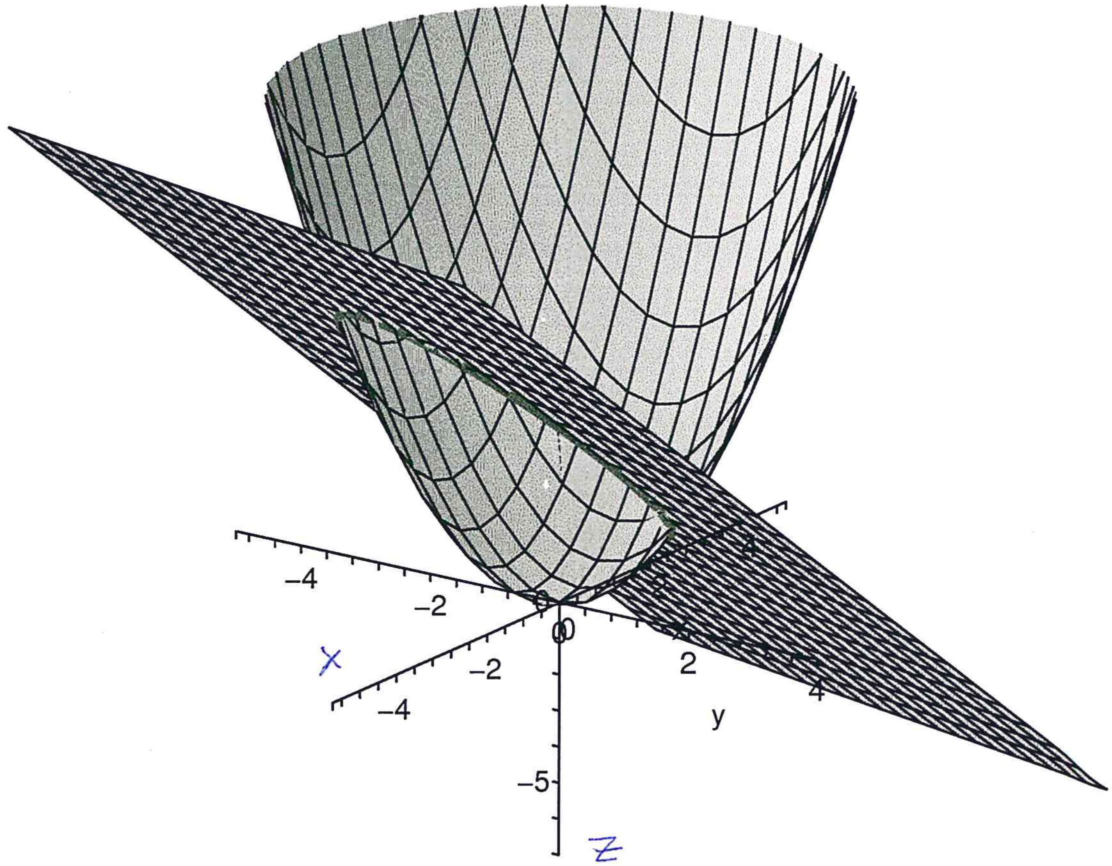
$$y = \frac{3}{\sqrt{2}}\sin t - \frac{1}{2} \qquad (0 \le t \le 2\pi).$$

Then $z = 4 - x - y = 4 - \left(\frac{3}{\sqrt{2}}\cos t - \frac{1}{2}\right) - \left(\frac{3}{\sqrt{2}}\sin t - \frac{1}{2}\right)$

$$= 5 - \frac{3}{\sqrt{2}}(\cos t + \sin t)$$

Thus,

$$r(t) = \left\langle \frac{3}{\sqrt{2}}\cos t - \frac{1}{2}, \frac{3}{\sqrt{2}}\sin t - \frac{1}{2}, 5 - \frac{3}{\sqrt{2}}(\cos t + \sin t) \right\rangle$$



−4
−2
0
x
−2
−4
0
2
4
y
2
4
−5
z

Ex Find a parametric equation for the line tangent to

$$r(t) = \langle t, t^2, t^3 \rangle \text{ at } t=1.$$

Sol $r'(t) = \langle 1, 2t, 3t^2 \rangle$. $r'(1) = \langle 1, 2, 3 \rangle$.

At $t=1$, $r(1) = \langle 1, 1, 1 \rangle$.

Let $\ell(t) = \langle 1, 2, 3 \rangle t + \langle 1, 1, 1 \rangle$.

$= \underline{\langle t+1, 2t+1, 3t+1 \rangle}$.

Ex Suppose $r'(t) = \langle t, \sin t, t^2+1 \rangle$ and $r(0) = \langle 1, 0, 2 \rangle$
Find $r(t)$.

Sol $r(t) = \int r'(t)\,dt = \langle \frac{1}{2}t^2 + C_1, -\cos t + C_2, \frac{1}{3}t^3 + t + C_3 \rangle$

$r(0) = \langle C_1, -1+C_2, C_3 \rangle$ needs to $= \langle 1, 0, 2 \rangle$.

Thus $C_1 = 1$, $C_2 = 1$, $C_3 = 2$.

We have

$$\underline{r(t) = \langle \tfrac{1}{2}t^2+1, 1-\cos t, \tfrac{1}{3}t^3+t+2 \rangle}.$$

Properties of Derivatives of vector valued functions.

1. $(u(t)+v(t))' = u'(t)+v'(t)$

2. $(c\,u(t))' = c\,u'(t)$

3. $(\underbrace{f(t)}_{\text{scalar}}\ \underbrace{u(t)}_{\text{vector}})' = f'(t)\,u(t) + f(t)\,u'(t)$

4. $(u(t)\cdot v(t))' = u'(t)\cdot v(t) + u(t)\cdot v'(t)$

5. $(u(t)\times v(t))' = u'(t)\times v(t) + u(t)\times v'(t)$

6. $(u(f(t)))' = u'(f(t))\,f'(t)$

Ex #75. Prove that $(r(t)\times r'(t))' = r(t)\times r''(t)$.

Proof $(r\times r')' = r'\times r' + r\times r'' = 0 + r\times r'' = r\times r''$.

Ex #76 Find an expression for $(u\cdot(v\times w))'$

Ans $(u\cdot(\cancel{v}\times w))' = u'\cdot(v\times w) + u\cdot(v\times w)' =$

$u'\cdot(v\times w) + u\cdot[v'\times w + v\times w'] =$

$u'\cdot(v\times w) + u\cdot(v'\times w) + u\cdot(v\times w')$.

Some proofs. (if time). The book proves #4.

Pf of 3 $\quad (f(t)\,u(t))' = f'(t)\,u(t) + f(t)\,u'(t)$.

$$(f(t)\langle u_1(t), u_2(t), u_3(t)\rangle)' = (\langle f u_1, f u_2, f u_3\rangle)'$$

$$= \langle (fu_1)', (fu_2)', (fu_3)' \rangle$$

$$= \langle f'u_1 + fu_1', f'u_2 + fu_2', f'u_3 + fu_3' \rangle$$

$$= \langle f'u_1, f'u_2, f'u_3\rangle + \langle fu_1', fu_2', fu_3' \rangle$$

$$= f'\langle u_1, u_2, u_3\rangle + f\langle u_1', u_2', u_3'\rangle$$

$$= f'u + fu'.$$

Ex 11 $\quad$ Important. Suppose $|r(t)| = c$, a constant.
Prove that $r \perp r'$.

Proof $\quad |r(t)|^2 = r \cdot r$ so $r \cdot r = c^2$

$$(r \cdot r)' = (c^2)' = 0$$

$$r' \cdot r + r \cdot r' = 0$$

$$r' \cdot r + r' \cdot r = 0$$

$$2(r' \cdot r) = 0$$

$$r' \cdot r = 0$$

Thus $r' \perp r$. Think circles, spheres.