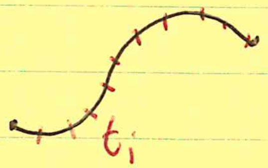
10.8

Arc Length and Curvature

Def distance = speed × time



Let $s_i = |r'(t_i)|$ = speed at time t_i

Then $\Delta L_i \approx s_i \Delta t_i$

Then arc length $L \approx \sum \Delta L_i \approx \sum s_i \Delta t_i$

In the limit as $\Delta t \to 0$ we have

$$L = \int_a^b |r'(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.$$

Ex Let $r(t) = \langle t, 2t, \frac{2}{3}t^{3/2} \rangle$, $0 \le t \le 1$. Find arc length.

Sol. $r'(t) = \langle 1, 2, t^{\frac{1}{2}} \rangle$. $|r'(t)| = \sqrt{1 + 4 + t} = \sqrt{5+t}$

Thus, $L = \int_0^1 \sqrt{5+t}\, dt$ Let $u = 5+t$. Then $du = dt$.

Now, $L = \int_5^6 u^{\frac{1}{2}}\, du = \frac{2}{3} u^{\frac{3}{2}} \Big|_5^6$

$$= \frac{2}{3}(6\sqrt{6} - 5\sqrt{5}) \approx 2.344399046$$

EX (This has extensive Calc II review; skip if time is short)

Let $r(t) = \langle t, 2t, t^2 \rangle$, $0 \le t \le 1$.

Find the length L.

Sol $r'(t) = \langle 1, 2, 2t \rangle$. $|r'(t)| = \sqrt{5+4t^2}$.

Thus, $L = \int_0^1 \sqrt{5+4t^2}\, dt$.

This integral is hard. We will use trig sub. (pg 322) and integration by parts (pg 321).

Trig sub Let $t = \frac{\sqrt{5}}{2}\tan\theta$. Then $dt = \frac{\sqrt{5}}{2}\sec^2\theta\, d\theta$.

Thus,

$$\sqrt{5+4t^2} = \sqrt{5+5\tan^2\theta} = \sqrt{5}\sqrt{1+\tan^2\theta} = \sqrt{5}\sec\theta.$$

[Note, since $0 \le t \le 1$, we have $0 \le \theta \le \tan^{-1}\left(\frac{2}{\sqrt{5}}\right) < \frac{\pi}{2}$. Thus, it is safe to assume $|\sec\theta| = \sec\theta$.]

Let $\theta_1 = \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$. Then

$$L = \int_0^{\theta_1} \sqrt{5}\sec\theta\, \frac{\sqrt{5}}{2}\sec^2\theta\, d\theta = \frac{5}{2}\int_0^{\theta_1} \sec^3\theta\, d\theta.$$

Int. by Parts $\quad \int \sec^3\theta\, d\theta = \int \underbrace{\sec\theta}\, \underbrace{\sec^2\theta\, d\theta} = uv - \int v\, du,$

Let $u = \sec\theta,\quad dv = \sec^2\theta\, d\theta.$

Then $du = \sec\theta\tan\theta\, d\theta\,,\quad v = \tan\theta$

$$= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\, d\theta,$$

$$= s\,t - \int (s^2 - 1)\, s\, d\theta,$$

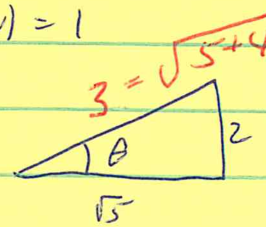
$$= s\,t - \int s^3\, d\theta + \int s\, d\theta.$$

Thus, $\quad 2\int s^3\, d\theta = s\,t + \ln|s + t|.$

Thus,

$$L = \frac{5}{2}\cdot\frac{1}{2}\left(s\,t + \ln|s+t|\right)\Big|_0^{\theta_1}.$$

To evaluate use: $\sec(0) = 1,\ \tan(0) = 1$



$\sec(\theta_1) = \sec\left(\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) = \frac{3}{\sqrt{5}}$

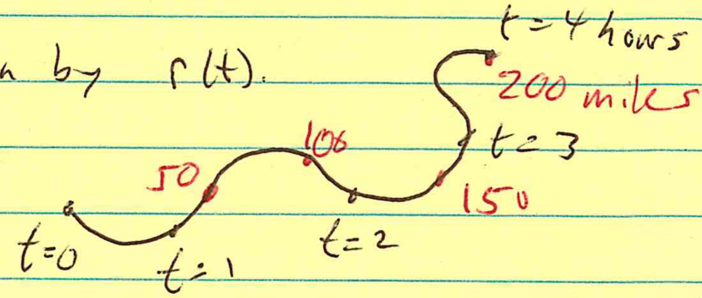
$\tan\left(\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) = 2/\sqrt{5}.$

Thus, $\quad L = \frac{5}{4}\left[\left(\frac{3}{\sqrt{5}}\frac{2}{\sqrt{5}} + \ln\left(\frac{3}{\sqrt{5}} + \frac{2}{\sqrt{5}}\right)\right) - \left(1\cdot 0 - \ln(1+0)\right)\right]$

$$= \frac{3}{2} + \frac{5}{4}\ln(\sqrt{5}) = \frac{3}{2} + \frac{5}{8}\ln(5) \approx 2.505898695$$

Arc Length as a "natural" parameter

Car trip is given by $r(t)$.



So, I can ask where you're at a given time, or when you traveled a given number of miles.

Let $\quad s(t) = \displaystyle\int_a^t |r'(t)|\, dt.$

Suppose we get $\quad s(t) = f(t)$. Then $t = f^{-1}(s)$

Thus we let

$$\hat{r}(s) = r(f^{-1}(s)).$$

This gives us position as a function of distance traveled.

Ex Let $r(t) = \langle 3t+1, 2t, t-2 \rangle$. Reparametrize in terms of arc length starting at $t=0$.

Sol $r' = \langle 3, 2, 1 \rangle$. $\quad |r'| = \sqrt{14}$. Thus $s = \sqrt{14}\, t$.

Thus $t = \frac{1}{\sqrt{14}} s$. Now

$$\hat{r}(s) = \left\langle \frac{3}{\sqrt{14}} s + 1,\ \frac{2}{\sqrt{14}} s,\ \frac{1}{\sqrt{14}} s - 2 \right\rangle.$$

Ex $r(t) = \langle t, 2t, \frac{2}{3}t^{\frac{3}{2}} \rangle$. Reparametrize in terms of arc length, starting at $t=0$.

Sol From before we have

Step 1 $$s(t) = \int_0^t \sqrt{5+t}\, dt = \frac{2}{3}(t+5)^{3/2} \Big|_0^t = \frac{2}{3}(t+5)^{3/2} - \frac{2}{3}(5)^{3/2}$$

Step 2 Solve for t as a function of s.

$$s = \frac{2}{3}(t+5)^{3/2} - \frac{10}{3}\sqrt{5}$$

$$\frac{3}{2}\left(s + \frac{10}{3}\sqrt{5}\right) = (t+5)^{3/2}$$

$$\left[\frac{3}{2}\left(s + \frac{10}{3}\sqrt{5}\right)\right]^{\frac{2}{3}} - 5 = t$$

$$t = \left[\frac{3}{2}s + 5\sqrt{5}\right]^{2/3} - 5$$

Step 3 Plug into $r(t)$.

$$\hat{r}(s) = r(t(s)) = \Big\langle \left[\frac{3}{2}s + 5\sqrt{5}\right]^{2/3} - 5,\ 2\left[\frac{3}{2}s + 5\sqrt{5}\right]^{2/3} - 10,$$
$$\frac{2}{3}\left(\left[\frac{3}{2}s + 5\sqrt{5}\right]^{\frac{2}{3}} - 5\right)^{3/2} \Big\rangle.$$

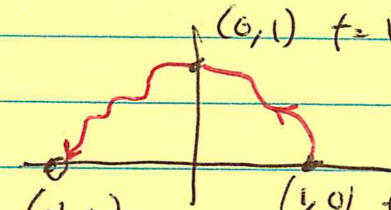
Done.

10.8 #12 is assigned. I will outline it, you do the details.

Let $r(t) = \left\langle \frac{2}{t^2+1} - 1, \frac{2t}{t^2+1} \right\rangle$. Reparameterize

in terms of arc length starting at $r_0 = \langle 1, 0 \rangle$.
What is the shape of this curve?

Sol. $r(t) = \langle 1, 0 \rangle$ when $t=0$. So we are starting at $t=0$.
Let's plot a couple of points just get a feel for $r(t)$.



Step 1 $s(t) = \int_0^t \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

simplify!

$s(t) = \int_0^t \frac{2}{t^2+1}\, dt = 2\arctan(t) - 0$

Step 2 Solve for t in terms of s. Easy.

Step 3 Plug into $r(t)$ to get

$\hat{r}(s) = \langle \hat{x}(s), \hat{y}(s) \rangle = \left\langle \frac{2}{\tan^2(\frac{s}{2}) + 1} - 1, \frac{2\tan(\frac{s}{2})}{\tan^2(\frac{s}{2}) + 1} \right\rangle$

Step 4 Simplify! Get

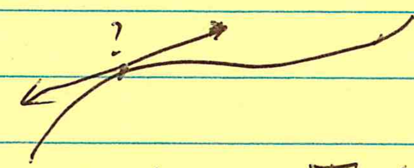
$$\hat{x}(s) = \text{something nice}$$
$$\hat{y}(s) = \text{something nice}$$

Then the shape will be obvious!

Unit tangent and Principal unit normal vectors, etc.

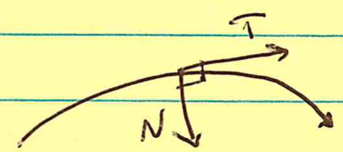
Given $r(t)$, let $T(t) = \dfrac{r'(t)}{|r'(t)|}$. Called <u>unit tangent vector</u>.

It is independent of the parameterization (provided $r'(t) \neq 0$) up to sign.



Since $|T| = 1$, we know $T \perp T'$.

Define $N(t) = \dfrac{T'(t)}{|T'(t)|}$. <u>Principal unit normal vector.</u>



Note: $N \cdot T = 0$

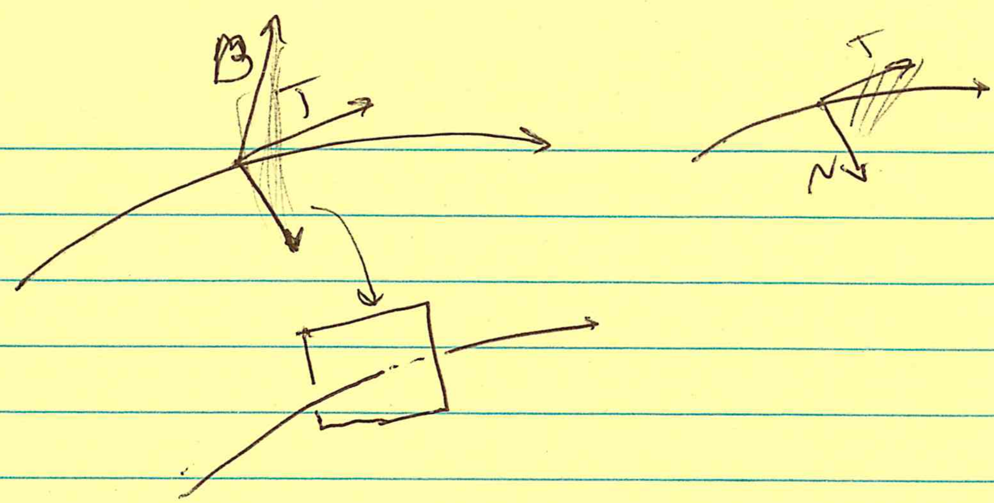
$N \cdot T = \dfrac{T' \cdot T}{|T'|} = 0.$

Define $B = T \times N$, the <u>binormal</u> <u>unit</u> <u>vector</u>.

Note: $|B| = |T \times N| = |T||N| \sin(\frac{\pi}{2}) = 1.$

The plane determined by T and N is called the <u>osculating plane</u>. The plane determined by N and B is called the <u>normal plane</u>.

<u>Note</u> For linear motion $N(t)$ is undefined since $T' = 0$.



Ex (#42, you have #41 for homework).

Let $r(t) = \langle t, t^2, t^3 \rangle$. Find equations for the N.P and the O.P. at $t=1$.

Sol N.P. is easy. We need a pt and a normal vector. We can use $r(1) = \langle 1,1,1 \rangle$ as our pt. We can use $T(1)$ for a normal vector. But $r'(1)$ works too and is easier.

$$r'(t) = \langle 1, 2t, 3t^2 \rangle, \quad r'(1) = \langle 1, 2, 3 \rangle.$$

N.P. $\quad \langle 1,2,3 \rangle \cdot (\langle x,y,z \rangle - \langle 1,1,1 \rangle) = 0.$

or $\quad x + 2y + 3z = 6$

The O.P. is determined by N and T. so $B = T \times N$ is a normal vector. But, here is a shortcut. In 10.9 we will learn that r' and r'' are in the O.P. Hence $r'(1) \times r''(1)$ will work as a normal vector.

$r''(t) = \langle 0, 2, 6t \rangle \quad r''(1) = \langle 0, 2, 6 \rangle.$

Let $n = r' \times r'' = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = \langle 6, -6, 2 \rangle.$

Or use $\langle 3, -3, 1 \rangle.$ $\left[B = \dfrac{r' \times r''}{|r' \times r''|} \right]$

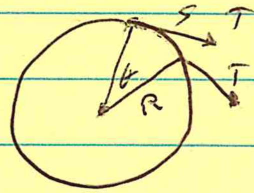
So now the O.P. is given by

$$\langle 3, -3, 1 \rangle \cdot \left(\langle x, y, z \rangle - \langle 1, 1, 1 \rangle \right) = 0$$

Or $3x - 3y + z = 1.$

Curvature $\quad \kappa = \left| \dfrac{dT(s)}{ds} \right|$ where s is arc length.

Discuss. Circle in xy-plane. Circle in Osc. P.



$\theta R = s \quad \theta = \dfrac{s}{R}$

$$\frac{d\theta}{ds} = \frac{1}{R} = \kappa.$$

It is hard to compute κ directly from this definition.

Thm [10] $\quad \kappa(t) = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$.

Ex $\quad r = \langle t, t^2, t^3 \rangle$. Find κ at $t = 1$.

Sol $\quad r'(1) = \langle 1, 2, 3 \rangle, \quad r''(1) = \langle 0, 2, 6 \rangle$.

$|r'(1)| = \sqrt{14}$.

$r' \times r'' = \langle 6, -6, 2 \rangle$. $\quad |r' \times r''| = \sqrt{36+36+4} = \sqrt{76}$

$= 2\sqrt{19}$.

Thus,

$$\kappa(1) = \frac{2\sqrt{19}}{14\sqrt{14}} = \frac{1}{7}\sqrt{\frac{19}{14}} \approx 0.166423535$$

∎

Proof of Thm 10.

First we show that $\kappa = \dfrac{|T'(t)|}{|r'(t)|}$.

Since $s = \int_a^t |r'(t)|\,dt$ we have $\dfrac{ds}{dt} = |r'(t)|$.

So, ~~by the chain rule~~, $\dfrac{dt}{ds} = \dfrac{1}{|r'(t)|}$

$$\frac{dT}{ds} = \frac{dT(t(s))}{ds} = \frac{dT}{dt}\frac{dt}{ds} = \frac{T'(t)}{|r'(t)|}.$$

Thus

$$\kappa = \left|\frac{dT}{ds}\right| = \frac{|T'(t)|}{|r'(t)|}.$$

Next,

$$T(t) = \frac{r'}{|r'|} = r'\frac{dt}{ds}$$

Thus, $\quad r' = \dfrac{ds}{dt}T(t) \Rightarrow r'' = s''T + s'T'$.

$$r' \times r'' = (s'T) \times (s''T + s'T') =$$

$$s's''\cancelto{0}{T \times T} + s's'\,T \times T'.$$

$\theta = \frac{\pi}{2}$

Now $T \perp T'$ so $|T \times T'| = |T||T'|\sin\theta = |T'|$.

Hence

$$|r' \times r''| = (s')^2|T'|. \text{ Thus } |T'| = \frac{|r' \times r''|}{(s')^2}$$

Or $|T'| = \dfrac{|r' \times r''|}{|r'|^2}$.

Finally,

$$k = \frac{|T'|}{|r'|} = \frac{|r' \times r''|}{|r'|^3} \text{ as claimed. } \boxed{/\!/}$$

__Curvature in rect. coordinates.__

__Thm__ Let $y = f(x)$. Then $k = \dfrac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}$

__Pf__ $r(t) = \langle t, f(t) \rangle$. Use $r(t) = \langle t, f(t), 0 \rangle$.

$r'(t) = \langle 1, f'(t), 0 \rangle$.

$r''(t) = \langle 0, f''(t), 0 \rangle$.

$$r' \times r'' = \begin{vmatrix} i & j & k \\ 1 & f' & 0 \\ 0 & f'' & 0 \end{vmatrix} = \langle 0, 0, f'' \rangle.$$

$|r' \times r''| = |f''(t)|$. $\quad |r'| = \sqrt{1+(f'(t))^2+0^2}$

$$k = \frac{|r' \times r''|}{|r'|^3} = \frac{|f''(t)|}{\left[1+(f'(t))^2\right]} \text{, but } t = x.$$

$\square$

Ex Let $y = \ln x$ for $x > 0$. (a) Find the maximum curvature. (b) Plot the function and the osculating circle at the point of maximum curvature.

(a) $y' = \frac{1}{x}$. $y'' = \frac{-1}{x^2}$. Thus, $\kappa(x) = \dfrac{\frac{1}{x^2}}{\left(1+\left(\frac{1}{x}\right)^2\right)^{3/2}}$.

Before computing $\kappa'(x)$ we simplify the expression for $\kappa(x)$.

$$\kappa(x) = x^{-2}\left(1+x^{-2}\right)^{-3/2} = \left(x^{\frac{4}{3}}\right)^{-\frac{3}{2}}\left(1+x^{-2}\right)^{-\frac{3}{2}}$$

$$= \left[x^{\frac{4}{3}}\left(1+x^{-2}\right)\right]^{-3/2} = \left[x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right]^{-\frac{3}{2}}$$

$$\kappa'(x) = -\frac{3}{2}\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right)^{-\frac{5}{2}}\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right)'$$

$$= -\frac{3}{2}\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right)^{-\frac{5}{2}}\left(\frac{4}{3}x^{\frac{1}{3}} - \frac{2}{3}x^{-\frac{5}{3}}\right)$$

Set $\kappa'(x) = 0$. Thus,

$$\frac{-2x^{\frac{1}{3}} + x^{-\frac{5}{3}}}{\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right)^{5/2}} = 0$$

The denominator cannot be 0. Therefore,

$$2x^{\frac{1}{3}} = x^{-\frac{5}{3}}$$

$$x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$, but only $+\frac{1}{\sqrt{2}}$ is in domain.

Therefore the max is at $x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

$$\kappa\left(\frac{\sqrt{2}}{2}\right) = \frac{2}{(1+2)^{3/2}} = \frac{2}{3\sqrt{3}} = \frac{2}{9}\sqrt{3} \approx 0.3849.$$

(b) The radius of the osculating circle is

$$R = \frac{1}{\kappa} = \frac{3\sqrt{3}}{2} \approx 2.598$$

The point of tangency is $\left(\frac{\sqrt{2}}{2}, \ln\left(\frac{\sqrt{2}}{2}\right)\right) = \left(\frac{\sqrt{2}}{2}, -\frac{\ln 2}{2}\right)$
$\approx (0.707, -0.347)$

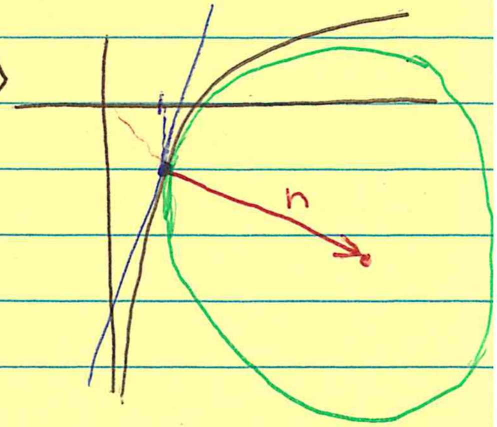
To find the center we construct a vector n of length R that is $\perp$ to the tangent line and pointing in the direction of curvature.

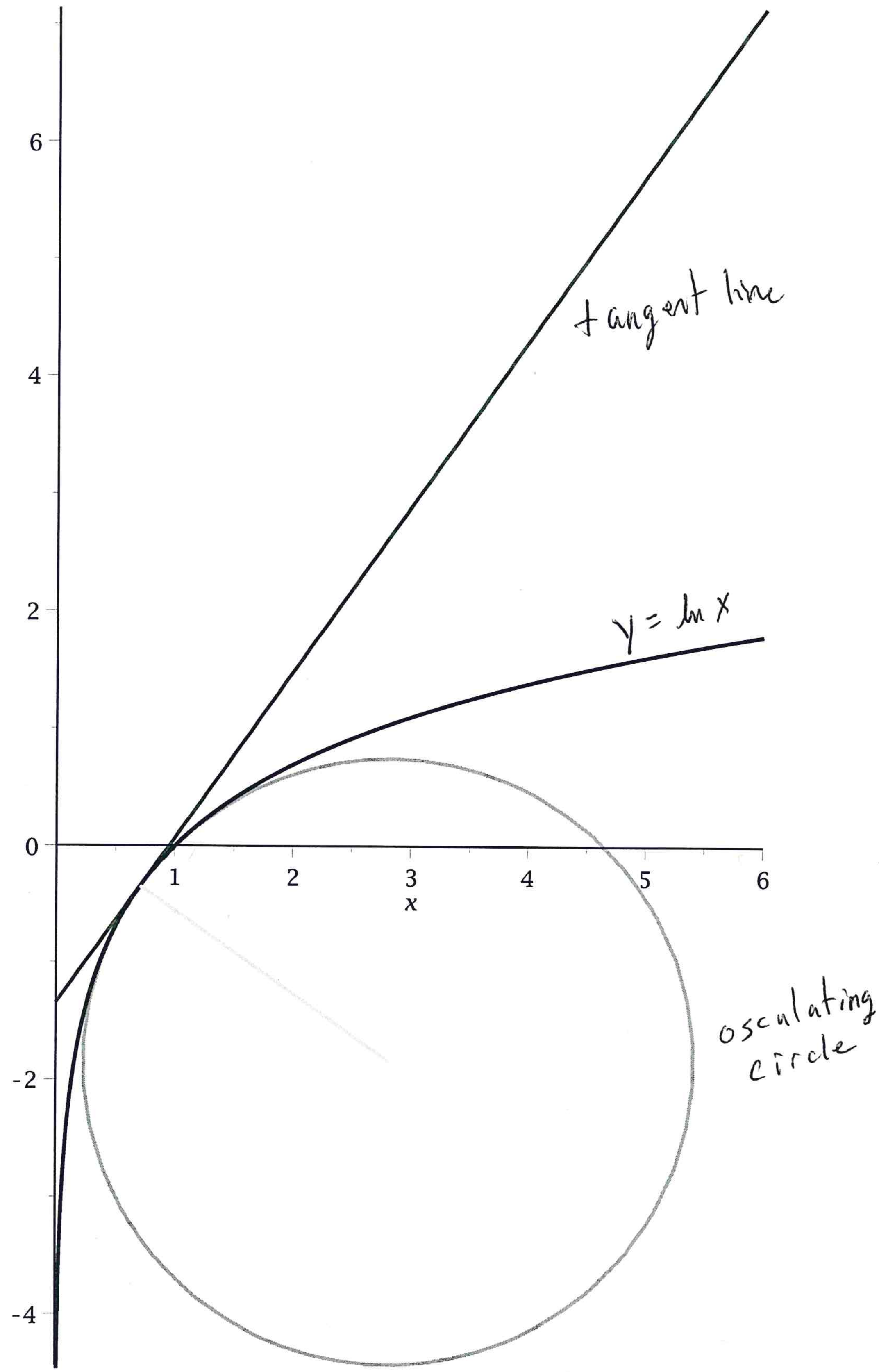
The slope of the tangent line is $(\ln x)' = \frac{1}{x}$ at $x = \sqrt{2}/2$. That is $\sqrt{2}$. The slope $\perp$ to this is $-1/\sqrt{2}$. We let

$$n = \frac{\langle \sqrt{2}, -1 \rangle}{|\langle \sqrt{2}, -1 \rangle|} \frac{3\sqrt{3}}{2} = \frac{\langle \sqrt{2}, -1 \rangle}{\sqrt{2+1}} \frac{3}{2}\sqrt{3} = \left\langle \frac{3\sqrt{2}}{2}, -\frac{3}{2} \right\rangle.$$

The center is $\left\langle \frac{\sqrt{2}}{2}, -\frac{\ln 2}{2} \right\rangle + \left\langle \frac{3\sqrt{2}}{2}, -\frac{3}{2} \right\rangle$

$$= \left\langle 2\sqrt{2}, -\frac{3+\ln 2}{2} \right\rangle.$$




tangent line
y = ln x
osculating circle
6
4
2
0
-2
-4
1
2
3
4
5
6
x