10.9 Motion: Velocity and Acceleration

$r(t) = \langle x(t), y(t), z(t) \rangle$ is position.
$v(t) = r'(t)$ is velocity $\quad v = |r'| = |v|$ is speed
$a(t) = r''(t)$ is acceleration.

$F = m a(t)$ is Force.

Ex An object has a mass of 5 kg, and $r(t) = \langle t, t^2, t^3 \rangle$.
Find $F(t)$.

Sol
$v(t) = \langle 1, 2t, 3t^2 \rangle$
$a(t) = \langle 0, 2, 6t \rangle$
$F(t) = 5a(t) = \langle 0, 10, 30t \rangle$.

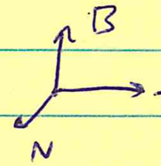
Ex An object has a mass of 6 kg and $F(t) = \langle 6, 6\sin t, 6\cos t \rangle$
and $v(0) = \langle 0, 0, 0 \rangle$, $r(0) = \langle 0, 1, 0 \rangle$.
Find $r(t)$.

Sol $a(t) = \dfrac{F(t)}{6} = \langle 1, \sin t, \cos t \rangle$

$v(t) = \langle t + C_1, -\cos t + C_2, \sin t + C_3 \rangle$
$= \langle t, 1 - \cos t, \sin t \rangle$.

$r(t) = \langle \frac{1}{2}t^2 + C_1, t - \sin t + C_2, -\cos t + C_3 \rangle$
$= \langle \frac{1}{2}t^2, 1 + t - \sin t, 1 - \cos t \rangle$.

Tangential and Normal Components of Acceleration.

<u>Goal</u> 

is a "natural" moving coordinate system for $\mathbf{a}(t)$. We want to write

$$\mathbf{a}(t) = a_T \mathbf{T} + a_N \mathbf{N} + \cancelto{0}{a_B \mathbf{B}}.$$

First, $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{v(t)}$ so, $\mathbf{v} = v\mathbf{T}$.

Thus $\mathbf{a} = \mathbf{v}' = (v\mathbf{T})' = v'\mathbf{T} + v\mathbf{T}'$.

Since $\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$ we have $\mathbf{T}' = |\mathbf{T}'|\mathbf{N}$. Thus,

$$\mathbf{a} = v'\mathbf{T} + v|\mathbf{T}'|\mathbf{N}$$

So far $a_T = v' = |\mathbf{r}'|'$, $a_N = v|\mathbf{T}'|$, $a_B = 0$.

But recall $\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v}$. So $|\mathbf{T}'| = v\kappa$.

Hence $a_N = v^2\kappa$ (discuss driving)

Now $\mathbf{a} = v'\mathbf{T} + v^2\kappa\mathbf{N}$.

It is handy to have formulas for a_T and a_N in terms of $\mathbf{r}(t)$.

$a_T = |\mathbf{r}'|'$, $a_N = v^2\kappa = |\mathbf{r}'|^2 \frac{|\mathbf{r}'\times\mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|\mathbf{r}'\times\mathbf{r}''|}{|\mathbf{r}'|}$

$\neq |\mathbf{r}''|$!!

There is another way to write a formula for a_T.
It involves a trick.

$$a = a_T T + a_N N \qquad v = v T.$$

$$v \cdot a = v T \cdot (a_T T + a_N N)$$

$$= v a_T \underbrace{T \cdot T}_{=1} + v a_N \underbrace{T \cdot N}_{=0}$$

$$= v a_T.$$

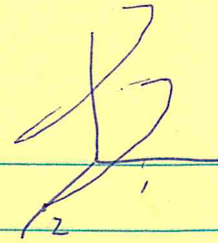
Thus $a_T = \dfrac{v \cdot a}{v} = \dfrac{r' \cdot r''}{|r'|}$.

$$\boxed{\begin{array}{ll} \underline{\text{Summary}} & a = a_T T + a_N N \\ \text{where} & a_T = v' = |r'|' = \dfrac{r' \cdot r''}{|r'|}. \\ & a_N = k v^2 = \dfrac{|r' \times r''|}{|r'|}. \end{array}}$$

Units $\quad a_N = \frac{m}{s^2} \qquad k = \frac{1}{m} \quad v = \frac{m}{s}.$

If $\quad a_N = (k)^p (v)^q \qquad q = 2$ and $p = 1$.

Ex Let $r(t) = \langle 2\cos t, \sin t, t \rangle$



For $t=\pi$, find $T(\pi)$, $N(\pi)$, K, a_T, a_N, eq of tangent line and the osc. plane.

Sol First find the "ingrediants".

$r(\pi) = \langle -2, 0, \pi \rangle$.

$r'(t) = \langle -2\sin t, \cos t, 1 \rangle \quad r'(\pi) = \langle 0, -1, 1 \rangle$

$r''(t) = \langle -2\cos t, -\sin t, 0 \rangle \quad r''(\pi) = \langle 2, 0, 0 \rangle$.

$|r'(t)| = \sqrt{4\sin^2 t + \cos^2 t + 1} \quad |r'(\pi)| = \sqrt{2}$.

$r'(\pi) \cdot r''(\pi) = 0 \quad |r'(\pi) \times r''(\pi)| = |\langle 0, 2, 2 \rangle| = 2\sqrt{2}$

$a_T = \dfrac{r' \cdot r''}{|r'|} = \dfrac{0}{\sqrt{2}} = 0 \qquad a_N = \dfrac{|r' \times r''|}{|r'|} = \dfrac{2\sqrt{2}}{\sqrt{2}} = 2$.

$K = \dfrac{|r' \times r''|}{|r'|^3} = \dfrac{2\sqrt{2}}{(\sqrt{2})^3} = 1$.

$T(\pi) = \dfrac{r'(\pi)}{|r'(\pi)|} = \left\langle 0, \dfrac{-1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle$.

But finding N requires finding $T'(t)$, so we cannot plug in $t=\pi$ yet.

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{4s^2+c^2+1}} \langle -2s, c, 1 \rangle$

Use scalar $\times$ vector product rule

$T'(t) = \left(\frac{1}{\sqrt{}}\right)' \langle \ , \ , \ \rangle + \frac{1}{\sqrt{}} \langle \ , \ , \ \rangle'$

$= -\frac{1}{2}(4s^2+c^2+1)^{-3/2}(\underbrace{8sc - 2cs}_{6sc} + 0)\langle -2s, c, 1\rangle +$

$\left(\frac{1}{\sqrt{}}\right)\langle -2c, -s, 0\rangle$

Now let $t = \pi$.

$T'(\pi) = -\frac{1}{2}(2)^{-\frac{3}{2}}(0)\langle 0, -1, 1\rangle + \frac{1}{\sqrt{2}}\langle 2, 0, 0\rangle$

$= \langle \sqrt{2}, 0, 0\rangle.$

Thus $N'(\pi) = \langle 1, 0, 0\rangle.$

t-line $\quad \ell(t) = \langle 0, -1, 1\rangle t + \langle -2, 0, \pi\rangle$

Osc-plane $\quad$ Need pt and normal vector.

Let $n = \langle 0, -1, 1\rangle \times \langle 1, 0, 0\rangle = \langle 0, 1, 1\rangle.$

$\langle 0, 1, 1\rangle \cdot (\langle x, y, z\rangle - \langle -2, 0, \pi\rangle) = 0$

$y + z = \pi$