11.2 Limits and Continuity

Def $\lim_{(x,y)\to(a,b)} f(x,y) = L$ means (roughly) that as (x,y) gets near to (a,b), the value of $f(x,y)$ approaches L.

More precisely, $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that
(for every) (there exists)

if $0 < \text{dist}\big((x,y),(a,b)\big) < \delta$

then $|f(x,y) - L| < \varepsilon$.

Def $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and equals $f(a,b)$.

Facts Sums, products and compositions of cont. functions are cont. So are quotients as long as the denominator is not zero.

Ex $x^3y + 3x^2y^2 + 7x$, $x^4y\sin(x+y^2)$, $\dfrac{x^2y+7y^3}{x^2+y^2+1}$,

y^3xe^{7xy} are continuous for all $(x,y) \in \mathbb{R}^2$.

Recall If $f(x)$ is undefined at $x=c$, but $f(x)=h(x)$ near c and $h(x)$ is cont at $x=c$, then

$$\lim_{x\to c} f(x) = h(c).$$

Ex $\displaystyle \lim_{x\to 1} \frac{x^2+x-2}{x^2+2x-3}$ appears to be "$\frac{0}{0}$" but it equals

$$\lim_{x\to 1} \frac{(x+2)(x-1)}{(x+3)(x-1)} = \lim_{x\to 1} \frac{x+2}{x+3} = \frac{3}{4}.$$

This also works for functions of two (or more) variables.

Ex Let $f(x,y) = \dfrac{x^3+xy^2+x^2+y^2}{x^2y+y^3-2x^2-2y^2}$. Find limit as $(x,y)\to(0,0)$.

Sol It appears that $f(0,0) =$ "$\frac{0}{0}$". But

$$f(x,y) = \frac{\cancel{(x^2+y^2)}(x+1)}{\cancel{(x^2+y^2)}(y-2)} \quad \text{for all } (x,y)\neq(0,0).$$

Thus,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x+1}{y-2} = -\frac{1}{2}.$$

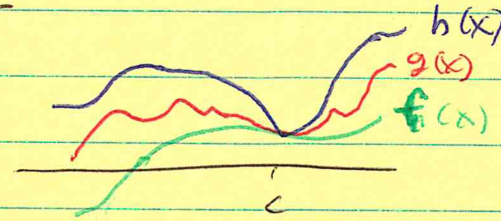
Recall: If $\lim_{x \to c} |f(x)| = 0$, then $\lim_{x \to c} f(x) = 0$.

This works for multivariable functions too.

Recall: The Squeeze Theorem: If $f(x) \le g(x) \le h(x)$ for x near c and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

then $$\lim_{x \to c} g(x) = L.$$



This also works for multivariable functions.

Ex: Let $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$. Find limit as $(x,y) \to (0,0)$.

Sol. $f(0,0) = \text{"}\frac{0}{0}\text{"}$ is undefined. But we notice

$$0 \le \frac{x^2}{x^2+y^2} \le 1 \quad \text{for all } (x,y) \neq (0,0).$$

Multiply through by $|y|$. Then

$$0 \le \left| \frac{x^2 y}{x^2+y^2} \right| \le |y|$$

As $(x,y) \to (0,0)$, clearly $|y| \to 0$. Thus, by the Sq Thm

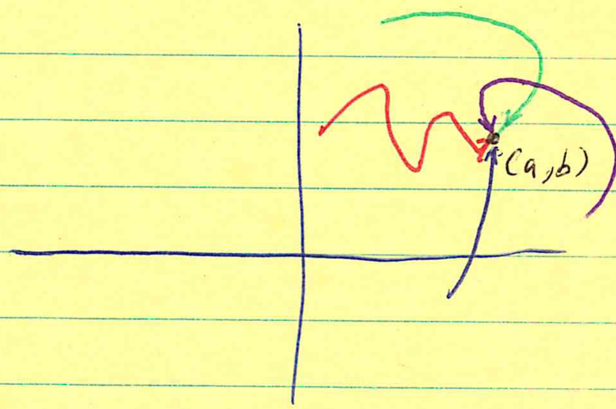
$$\lim_{(x,y) \to (0,0)} \left| \frac{x^2 y}{x^2+y^2} \right| = 0.$$

Thus $\lim_{(x,y) \to (0,0)} f(x,y) = 0$. $\square$

Recall If $\lim_{x \to c} f(x)$ exists, then $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$.

That is, the limits from the left and right must match.

With multivariable functions there are many more paths that (x,y) could travel along the way to (a,b). Limits along all these paths must exist and match (be equal) in order for limit $(x,y) \to (a,b)$ to exist.



We can use this to show that certain limits do not exist.

Ex Let $f(x,y) = \dfrac{x^2}{x^2+y^2}$. Show limit $(x,y) \to (0,0)$ does not exist.

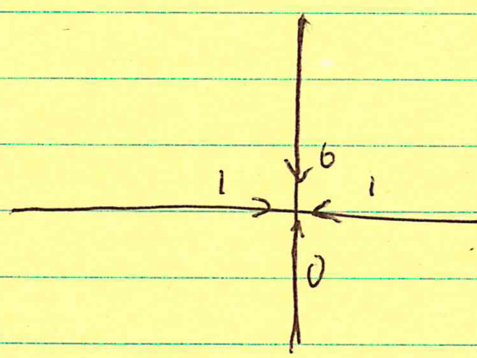
Sol Hold $x=0$ and take the limit as $y \to 0^{\pm}$.
Since $f(0,y)=0$ we have

$$\lim_{y \to 0} f(0,y) = \lim_{y \to 0} 0 = 0.$$

But now hold $y=0$ and take the limit as $x \to 0^{\pm}$.
Now

$$f(x,0) = \frac{x^2}{x^2+0^2} = 1.$$

Thus $\displaystyle\lim_{x \to 0} f(x,0) = \lim_{x \to 0} 1 = 1.$



These limit do not match, $0 \neq 1$, hence

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} \quad \text{does not exist.}$$

There exist functions where every linear path gives the same ~~value for~~ limiting value, but there are curved paths that don't match. ~~these~~.

Ex Let $f(x,y) = \dfrac{x^3 y}{x + y^3}$.

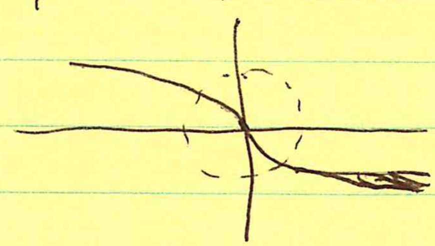
Clearly,

$$\lim_{y \to 0} f(0,y) = 0 \qquad \lim_{x \to 0} f(x,0) = 0.$$

In fact if we let $y = mx$ and take $x \to 0$, we get

$$f(x, mx) = \frac{m x^4}{x + m^3 x^3} = \frac{m x^3}{1 + m^3 x^2} \longrightarrow 0 \text{ as } x \to 0.$$

Yet $f(x,y)$ is not even defined along the curve $x = -y^3$. So, the limit d.n.e.



These ideas carry ~~othe~~ over to functions of three variables

<u>Ex</u> $f(x,y,z) = \dfrac{xy+yz+xz}{3x^2+2y^2+z^2}$. Study limit as $(x,y,z)\to(0,0,0)$

Along any axis, the limit is zero. For example Let $x=y=0$ and take the limit as $z\to 0$.

$$\lim_{z\to 0} f(0,0,z) = \lim \frac{0}{z^2} = 0$$

But take the limit along the line $x=y,\ z=0$.

$$\underset{x\to 0}{\text{limit}}\ f(x,x,0) = \lim \frac{x^2}{3x^2} = \frac{1}{5}$$

Take the limit along the line $x=y=z$.

$$\lim_{x\to 0} \frac{3x^2}{6x^2} = \frac{1}{2}.$$

Thus the general limit d.n.e.

Ex Use polar coordinates to show that

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$$

Sol $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$.

Thus $$\frac{3x^2y}{x^2+y^2} = \frac{3r^3\cos^2\theta\sin\theta}{r^2} = 3r\cos^2\theta\sin\theta$$

Since $-1 \le \cos^2\theta\sin\theta \le 1$ we know

$$-3r \le 3r\cos^2\theta\sin\theta \le 3r.$$

As $(x,y)\to(0,0)$, $r\to 0$. Thus, by the ~~sq~~ Squeeze Theorem

$$\lim_{r\to 0} 3r\cos^2\theta\sin\theta = 0.$$

This is the same as

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$$

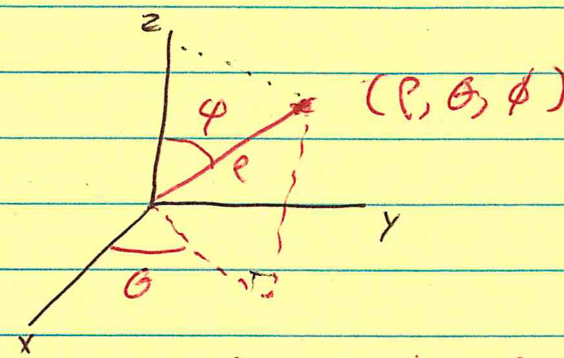
[Optional Reading]

Ex Use spherical coordinate to show that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0.$$

Sol We have not covered spherical coordinates yet so this is <u>optional reading</u>.

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$
$$\rho = x^2+y^2+z^2$$



(ρ = rho, the Greek letter)

Now

$$\frac{xyz}{x^2+y^2+z^2} = \frac{\rho^3 \sin^2\phi \cos\theta \sin\theta \cos\phi}{\rho^2}$$

$$= \rho \sin^2\phi \cos\theta \sin\theta \cos\phi.$$

Since $-1 \le \sin^2\phi \cos\theta \sin\theta \cos\phi \le 1$ we know

$$-\rho \le \rho \sin^2\phi \cos\theta \sin\theta \cos\phi \le \rho.$$

Thus the limit as $\rho \to 0$ of the middle term is 0 by The Squeeze Thm. Hence,

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0.$$