11.3 Partial Derivatives

Def The partial derivative of $f(x,y)$ with respect to x at (x_0, y_0) is

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} f(x_0 + \Delta x, y_0).$$

The partial derivative of $f(x,y)$ with respect to y at (x_0, y_0) is

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} f(x_0, y_0 + \Delta y).$$

In practice this is easy, you just treat the other variable as a constant.

Ex Let $f(x,y) = x^2 y^3$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Sol $\frac{\partial f}{\partial x} = 2xy^3$. $\quad \frac{\partial f}{\partial y} = 3x^2y^2$.

Ex Let $f(x,y) = \sin(xy)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Sol

$$\frac{\partial f}{\partial x} = \cos(xy) \cdot \frac{\partial xy}{\partial x} = \cos(xy) \cdot y = y\cos(xy).$$

$$\frac{\partial f}{\partial y} = \cos(xy) \frac{\partial xy}{\partial y} = \cos(xy) \cdot x = x\cos(xy).$$

Ex Let $f(x,y) = xy\cos(xy^2)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Sol.
$$\frac{\partial f}{\partial x} = y\cos(xy^2) - xy\sin(xy^2)\underbrace{\frac{\partial xy^2}{\partial x}}_{= y^2}$$

$$= y\cos(xy^2) - xy^3\sin(xy^2)$$

$$\frac{\partial f}{\partial y} = x\cos(xy^2) - xy\sin(xy^2)\underbrace{\frac{\partial xy^2}{\partial y}}_{= 2xy}$$

$$= x\cos(xy^2) - 2x^2y^2\sin(xy^2).$$

Notation
$$\frac{\partial f}{\partial x} = \partial_x f = f_x = \partial_1 f$$

$$\frac{\partial f}{\partial y} = \partial_y f = f_y = \partial_2 f$$

Higher Order Partial Derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \partial_{xx} f = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \partial_{xy} f = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3} = \partial_{xxx} f = f_{xxx}$$

And so on...

Examples

Let $f(x,y) = xy^2 + \sin(xy)$.

$$f_x = y^2 + y\cos(xy)$$

$$f_{xx} = -y^2\sin(xy)$$

$$f_{xy} = 2y + \cos(xy) - xy\sin(xy)$$

$$f_y = 2xy + x\cos(xy)$$

$= !$

$$f_{yy} = 2x - x^2\sin(xy)$$

$$f_{yx} = 2y + \cos(xy) - xy\sin(xy).$$

Thm If f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$.

The proof is in Appendix D, pg A35, and uses the Mean Value Thm.

All this works for functions of 3 or more variables.

Ex Let $f(x,y,z) = xyz + x^2 \sin(yz)$.

$\partial_z f = xy + x^2 y \cos(yz)$

$\partial_x f = yz + 2x \sin(yz)$

$\partial_{xy} f = z + 2xz \cos(yz)$

$\partial_{zy} f = x + x^2 \cos(yz) - x^2 yz \sin(yz)$

$f_{xyz} = 1 + 2x \cos(yz) - 2xyz \sin(yz)$

$f_{zz} = -x^2 y^2 \sin(yz)$

$f_{zzx} = -2xy^2 \sin(yz)$

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Solution

$$u = (x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$u_x = -\tfrac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot(2x) = -x(\qquad)^{-3/2}$$

$$u_{xx} = -(\qquad)^{-3/2} + \tfrac{3}{2}x(\qquad)^{-5/2}(2x)$$

$$= -(\qquad)^{-3/2} + 3x^2(\qquad)^{-5/2}$$

Likewise, $u_{yy} = -(\qquad)^{-3/2} + 3y^2(\qquad)^{-5/2}$

and $u_{zz} = -(\qquad)^{-3/2} + 3z^2(\qquad)^{-5/2}$.

Thus, $u_{xx}+u_{xy}+u_{zz} = -3(\qquad)^{-3/2} + 3(x^2+y^2+z^2)(\qquad)^{-5/2}$

$$= -3(\qquad)^{-3/2} + 3(\qquad)^{-3/2}$$

$$= 0. \checkmark$$

__Note:__ This is not valid for $x=y=z=0$.

__Note:__ Functions with this property are called __harmonic functions__ and have many applications.