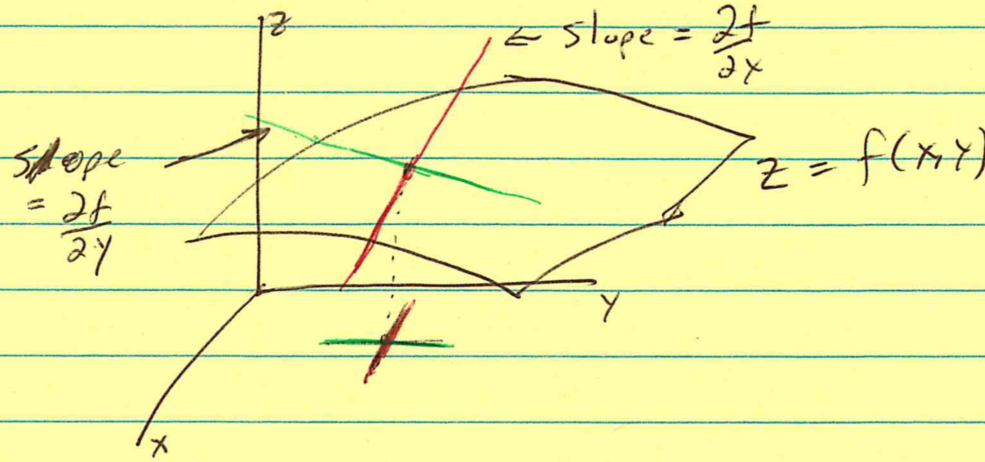
<u>11.4</u> Tangent Planes and Linear Approximations.

We can think about partial derivatives geometrically.



$\frac{\partial f}{\partial x}$ is the slope of a line tangent to the surface that is parallel to the xz-plane.

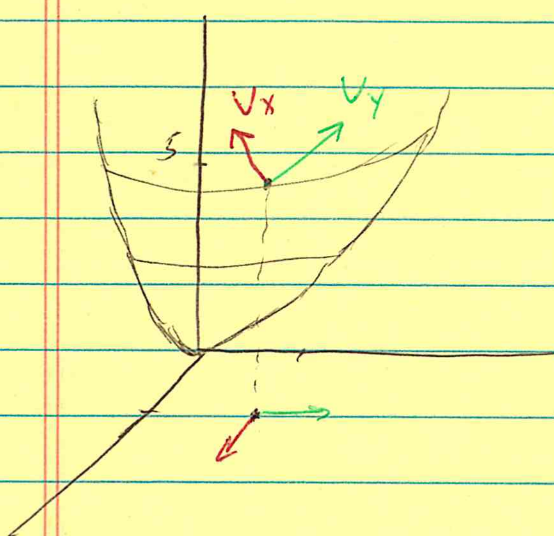
$\frac{\partial f}{\partial y}$ is the slope of a line tangent to the surface that is parallel to the yz-plane.

These two lines determine a plane called the <u>tangent plane</u>.

To find a equation for the tangent plane we select vectors in these two lines and take their cross product to get a normal vector.

Ex Let $z = f(x,y) = x^2 + \frac{y^2}{4}$. Find an equation for the plane tangent to this surface at $(2,2,5)$.

Sol



$$\frac{\partial f}{\partial x} = 2x = 4 \text{ when } x = 2$$

$$\frac{\partial f}{\partial y} = \frac{y}{2} = 1 \text{ when } y = 2.$$

Let $v_x = \langle 1, 0, 4 \rangle$.

Let $v_y = \langle 0, 1, 1 \rangle$.

Then let $n = v_x \times v_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 4 \\ 0 & 1 & 1 \end{vmatrix} = \langle -4, -1, 1 \rangle$.

The tangent plane is given by

$$n \cdot (\langle x, y, z \rangle - \langle 2, 2, 5 \rangle) = 0$$

$$-4(x-2) - (y-2) + (z-5) = 0$$

$$-4x - y + z = -5$$

or

$$\boxed{4x + y - z = 5}$$

or

$$\boxed{z = 5 - 4x - y}$$

Ex Let $z = f(x, y) = x^2 y$. Find the tangent plane when $x_0 = 1$, $y_0 = 2$.

Sol. $\partial_x f = 2xy \quad \partial_x f(1,2) = 4.$

$\partial_y f = x^2 \quad \partial_y f(1,2) = 1.$

$z_0 = f(x_0, y_0) = f(1,2) = 2.$ Let $p_0 = \langle 1, 2, 2 \rangle$.

$$n = \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$$

$$= \langle -f_x, -f_y, 1 \rangle = \langle -4, -1, 1 \rangle.$$

The tangent plane is given by

$$n \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0$$

$$\vdots$$

$$\boxed{4x + y - z = 4}$$

or

$$\boxed{z = 4x + y - 4}$$

See the video I posted.

Linear Approx. with tangent planes.

Ex Use the tangent plane to approximate $f(1.1, 1.9)$ $(f(x,y) = x^2 y)$.

Sol Near $(1,2)$ we showed $f(x,y) \approx 4x + y - 4$.

Thus, $f(1.1, 1.9) \approx 4(1.1) + 1.9 - 4 = 4.4 + 1.9 - 4.0 = 2.3$

Note, $f(1.1, 1.9) = (1.1)^2(1.9) = 2.299$.

Linear Approx using differentials.

Differentials are just an older notation for linear approximations. The 1-dimensional version was covered in Section 2.8. Let $y = f(x)$. Let "dx" be a small change in x. Then the change in y is approximately

$$dy = f'(x)\, dx.$$

Ex Suppose $y = f(x) = x^2$. Approximate $f(1.1)$.

Sol. $f(1) = 1^2 = 1$. $f'(1) = 2(1) = 2$. The change in y is approx. $dy = f'(1)(0.1) = 0.2$. Thus

$$y \approx f(1) + dy = 1.2.$$

(The exact value is $f(1.1) = (1.1)^2 = 1.21$.)

For functions of two variables ~~it is how~~ it is similar.
Let $z = f(x,y)$. Suppose "dx" and "dy" are small changes in the values of x and y, resp. Then the change in z is approx.

$$dz = f_x\,dx + f_y\,dy.$$

Ex Let $f(x,y) = x^2 y$. Approximate $f(1.1, 1.9)$.

Sol. $f(1,2) = 2$. Let $dx = 0.1$ and $dy = -0.1$.
~~Then~~ Now $f_x = 2xy = 2(1)(2) = 4$ and
$f_y = x^2 = (1)^2 = 1$.
Thus,

$$dz = (4)(0.1) + (1)(-0.1) = 0.3.$$

So $f(1.1, 1.9) \approx f(1,2) + dz = 2.3$,

just as before.

Differentials are just another way of using the tangent plane to do linear approximations.