11.5 The Chain Rule

Let $f(x,y)$ be given, but suppose x and y are functions of another variable t. Then we may want to know how does $f(x(t), y(t))$ change with respect to t. We need a new version of the chain rule.

As a first example suppose $f(x,y)$ is a plane,

$$z = f(x,y) = ax + by + c.$$

Then $$\frac{dz}{dt} = a\frac{dx}{dt} + b\frac{dy}{dt} + 0$$

Notice that $a = \frac{\partial z}{\partial x}$ and $b = \frac{\partial z}{\partial y}$.

In the more general case, suppose $f(x,y)$ is smooth enough that it is well approximated by a tangent plane. Suppose we want to find the rate of change of f when $t = t_0$ and that $x_0 = x(0)$, $y_0 = y(0)$. The equation of the tangent plane here is

$$z = \underbrace{\left(\frac{\partial f}{\partial x}\right)_{t=t_0}}_{\text{There are constants}} x + \underbrace{\left(\frac{\partial f}{\partial y}\right)_{t=t_0}}_{\text{There are constants}} y + \underbrace{\left[f(x_0,y_0) - \left(\frac{\partial f}{\partial x}\right)_{t=t_0} x_0 - \left(\frac{\partial f}{\partial y}\right)_{t_0} y_0\right]}_{\text{There are constants}}$$

There are constants

Then $$\frac{df}{dt} = \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$ (The Chain Rule)

Your textbook gives a proof that this is correct.

Ex Let $f(x,y) = x^2 + xy$, $x = t^2$ and $y = \sin t$. Find df/dt.

Sol. 1

$$\frac{df}{dt} = f_x x' + f_y y' = (2x+y)(2t) + x\cos t$$

$$= (2t^2 + \sin t)(2t) + t^2\cos t$$

$$= 4t^3 + 2t\sin t + t^2\cos t.$$

Sol 2 We do it the Calc I way for comparison.

$$f(x,y) = x^2 + xy = t^4 + t^2\sin t.$$

Thus, $\frac{df}{dt} = 4t^3 + 2t\sin t + t^2\cos t.$

Note Here is something to notice that will come up later

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle.$$

Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. Define $\nabla f = \langle f_x, f_y \rangle$; this is the gradient of f, also denoted grad f.

Now we can write

$$\frac{d(f(\mathbf{r}(t)))}{dt} = \nabla f \cdot \mathbf{r}'(t)$$

Some variations.

Ex Let $f(t,y) = t^2 + ty^3$ with $y = e^{2t}$. Then

$$\frac{df}{\partial t} = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (2t+y^3)\cdot 1 + 3ty^2 \cdot 2e^{2t}$$

$$= 2t + y^3 + 6ty^2e^{2t}$$

$$= 2t + e^{6t} + 6te^{4t}e^{2t}$$

$$= 2t + (1+6t)e^{6t}.$$

Another version of the Chain Rule is for cases like this $f(x(s,t), y(s,t))$. Then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Ex Let $f(x,y) = x^2y$, $x = \sin(s+t)$, $y = \cos(s+t)$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = 2xy\cos(s+t) + x^2(-\sin(s+t))$$

$$= 2xy^2 - x^3.$$

$\frac{\partial f}{\partial t} =$ The same in this case.

This extends to three or more variables

Ex Let $f(x,y,z) = xyz^2$ with $x = st$ $y = s^2t$ $z = \sin t$.

Find $\frac{\partial f}{\partial s}$.

Solution $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$

$$= yz^2t + xz^2 2st + 0$$

$$= s^2t^2 \sin^2 t + 2s^2t^2 \sin^2 t = 3s^2t^2 \sin^2 t$$

Ex Let $f(x,y) = x^2 + 3y$. Find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$.

Sol. Recall $x = r\cos\theta$ and $y = r\sin\theta$.

Note that $x_r = \cos\theta$, $x_\theta = -y$, $y_r = \sin\theta$, $y_\theta = x$.

Then $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial f}{\partial r} = 2x\cos\theta + 3\sin\theta$

$$= 2r\cos^2\theta + 3\sin\theta$$

And

$$\frac{\partial f}{\partial \theta} = f_x x_\theta + f_y y_\theta = -2xy + 3x$$

$$\frac{2x^2}{\sqrt{x^2+y^2}} + \frac{3y}{\sqrt{x^2+y^2}}$$

#42 (#45 is similar and is assigned.)

Let $u=f(x,y)$, $x=e^s\cos t$, $y=e^s\sin t$.

Prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=e^{-2s}\left[\dfrac{\partial^2 u}{\partial s^2}+\dfrac{\partial^2 u}{\partial t^2}\right]$,

or $u_{xx}+u_{yy}=e^{-2s}(u_{ss}+u_{tt})$.

Proof Note that $\frac{\partial x}{\partial s}=x$, $\frac{\partial y}{\partial s}=y$, $\frac{\partial x}{\partial t}=-y$, $\frac{\partial y}{\partial t}=x$.

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}=u_x x+u_y y$$

$$\frac{\partial^2 u}{\partial s^2}=\frac{\partial}{\partial s}(u_x x+u_y y)=\frac{\partial u_x}{\partial s}x+u_x\frac{\partial x}{\partial s}+\frac{\partial u_y}{\partial s}y+u_y\frac{\partial y}{\partial s}$$

$$=(u_{xx}x+u_{yx}y)x+u_x x+(u_{yx}x+u_{yy}y)y+u_y y$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}=-u_x y+u_y x$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial t}(-u_x y+u_y x)=-\frac{\partial u_x}{\partial t}y-u_x\frac{\partial y}{\partial t}+\frac{\partial u_y}{\partial t}x+u_y\frac{\partial x}{\partial t}$$

$$-(-u_{xx}y+u_{xy}x)y-u_x x+(-u_{yx}y+u_{yy}x)x-u_y y$$

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = u_{xx} x^2 + u_{yx} y x + u_x x + u_{yx} x y + u_{yy} y^2 + u_y y$$

$$+ u_{xx} y^2 - u_{xy} x y - u_x x - u_{yx} y x + u_{yy} x^2 - u_y y$$

$$= u_{xx}(x^2+y^2) + u_{yy}(x^2+y^2)$$

$$= (x^2+y^2)(u_{xx}+u_{yy})$$

$$x^2+y^2 = e^{2s}\cos^2 t + e^{2s}\sin^2 t = e^{2s}$$

Thus,

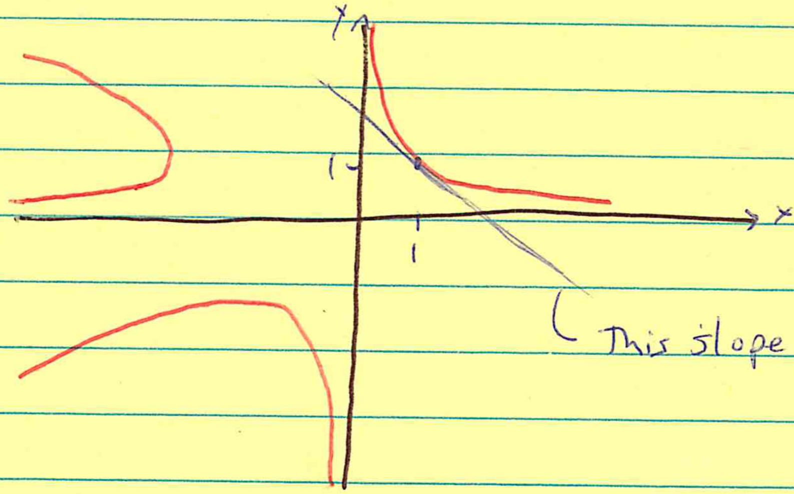
$$u_{xx}+u_{yy} = e^{-2s}(u_{ss}+u_{tt}).$$

11.5 The Chain Rule and Implicit Differentiation.

You did a version of this in Calc I. Here is an example.

Ex Let $xy^3 + x^2y = 2$. If we change x then y will have will to change if the relation is to hold. Thus we can regard y as an implicit function of x. (If we could write y as an expression in x, then y would be a explicit function of x

Consider the point $(1,1)$ on the graph of $xy^3 + x^2y = 2$.



This slope is $\frac{dy}{dx}(1,1)$.

Find $\frac{dy}{dx}$ at this point.

Solution Apply $\frac{d}{dx}$ to both sides:

$$\frac{d}{dx}(xy^3 + x^2y) = \frac{d\,2}{dx}$$

$$\underbrace{\frac{dx}{dx}}_{=1} y^3 + x\,3y^2 \underline{\frac{dy}{dx}} + 2x \underbrace{\frac{dx}{dx}}_{=1} + x^2 \underline{\frac{dy}{dx}} = 0$$

This is what we seek

$$y^3 + 2x + 3xy^2 \frac{dy}{dx} + x^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-y^3 - 2x}{3xy^2 + x^2} = \frac{-1-2}{3+1} = \frac{-3}{4}$$

New! Here is a new method. Let $f(x,y) = xy^3 + x^2y$ and set $f(x,y) = 2$.

$$\frac{d\,f(x, y(x))}{dx} = \frac{d\,2}{dx} = 0$$

$$\shortparallel$$

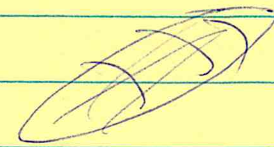
$$\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-f_x}{f_y} = \frac{-(y^3 + 2xy)}{3xy^2 + x^2} = \frac{-3}{4}$$

The same idea works with three variables.

Ex Consider the ellipsoid,

$$x^2 + 4y^2 + 9z^2 = 41.$$



Consider the point $(1,1,2)$ on it.
If we hold x fixed and make a small change to y, how will z have to change if we are to stay on this ellipsoid?

That is, what is $\frac{\partial z}{\partial y}(1,1)$?

(We are regarding z as an implicit function of x and y.)

Sol Let $F(x,y,z) = x^2 + 4y^2 + 9z^2$ and set $F = 41$.

$$\frac{\partial F}{\partial y} = 0 + 8y \cdot 1 + 18z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-8y}{18z} = \frac{-8}{18 \cdot 2} = \frac{-2}{9} \qquad \square$$

In general if $F(x,y,z)=c$ we can regard z as a function of x and y and compute $\frac{\partial z}{\partial x}$ and $\partial z/\partial y$ as follows.

$\boxed{\frac{\partial z}{\partial x}}$

$$\frac{\partial F}{\partial x} = \frac{\partial c}{\partial x}$$

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

Solve for this.

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$

$\boxed{\frac{\partial z}{\partial y}}$

$$\frac{dF}{dy} = \frac{dc}{dy}$$

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial y}}_{=0} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial y}}_{=1} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0$$

Find this.

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

Ex Suppose $Q(\theta, h, r, A) = 7$.

If θ and r are held fixed, but A changes, how does h change if the quantity Q is held at 7?

Sol We need to find a formula for $\frac{\partial h}{\partial A}$.

$$\frac{dQ}{dA} = \frac{d7}{dA}$$

$$\frac{\partial Q}{\partial \theta}\underbrace{\frac{\partial \theta}{\partial A}}_{=0} + \frac{\partial Q}{\partial h}\underbrace{\frac{\partial h}{\partial A}}_{\text{Find this!}} + \frac{\partial Q}{\partial r}\underbrace{\frac{\partial r}{\partial A}}_{=0} + \frac{\partial Q}{\partial A}\underbrace{\frac{\partial A}{\partial A}}_{=1} = 0$$

$$\frac{\partial h}{\partial A} = \frac{-\frac{\partial Q}{\partial A}}{\frac{\partial Q}{\partial h}} = \frac{-Q_A}{Q_h}$$

Ex Suppose $Q(\theta, h, r, A) = 7$.

If θ is held fixed, but A changes, how does h change if Q is held at 7?

Sol Since we do not know how r is or is not affected by changing A we can only do the following

$$\frac{\partial Q}{\partial \theta}\underbrace{\frac{\partial \theta}{\partial A}}_{0} + \frac{\partial Q}{\partial h}\underbrace{\frac{\partial h}{\partial A}}_{\text{Find this}} + \frac{\partial Q}{\partial r}\underbrace{\frac{\partial r}{\partial A}}_{= ??} + \frac{\partial Q}{\partial A}\underbrace{\frac{\partial A}{\partial A}}_{= 1} = 0$$

$$\frac{\partial h}{\partial A} = \frac{-\frac{\partial Q}{\partial A} - \frac{\partial Q}{\partial r}\frac{\partial r}{dA}}{\frac{\partial Q}{\partial h}} = \frac{-Q_A - Q_r \frac{\partial r}{dA}}{Q_h}.$$

Without more info. this is the best we can do.