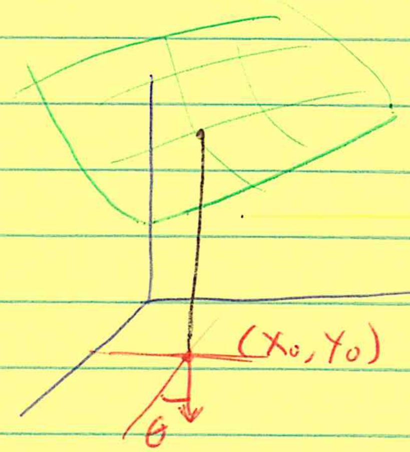
11.6 Directional Derivatives and The Gradient Vector

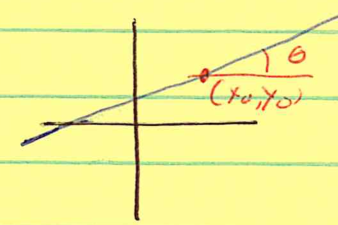
Idea



How fast does z change if we move away from (x_0, y_0) in the direction θ (angle from positive x direction)?

We write parametric equations for a line in the xy-plane that goes through (x_0, y_0) when $t = 0$ and makes an angle θ with the positive x-direction.

$$x(t) = (\cos\theta)t + x_0$$
$$y(t) = (\sin\theta)t + y_0$$



Let $z(t) = f(x(t), y(t))$. Then

$$\frac{dz}{dt} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x \cos\theta + f_y \sin\theta$$

$$= \langle f_x, f_y \rangle \cdot \langle \cos\theta, \sin\theta \rangle$$

$$= \nabla f \cdot u_\theta \qquad = D_\theta f$$

The gradient of f.

unit vector in the direction θ.

The directional derivative of f in the direction θ.

Ex Let $f(x,y) = x^3y + 3xy + 2y$. Find the rate of change of f at the point $(1,2)$ moving in the direction 60° wrt the $+x$ direction.

Sol $D_{60^\circ} f(1,2) = \nabla f \cdot u_\theta = \langle f_x, f_y \rangle \cdot \langle \cos 60^\circ, \sin 60^\circ \rangle$

$$= \langle 3x^2y + 3y,\ x^3 + 3x + 2 \rangle \cdot \langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle$$

$$= \langle 12, 6 \rangle \cdot \langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle = 6 + 3\sqrt{3} \approx 11.196$$

Ex Find $D_{30^\circ} f(1,2)$, for the same function.

Sol $= \langle 12, 6 \rangle \cdot \langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2} \rangle = 6\sqrt{3} + 3 \approx 13.392$

Ex Find $D_{0^\circ} f(1,2)$, for the same function.

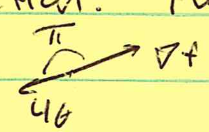
Sol $D_{0^\circ} f(1,2) = \langle 12, 6 \rangle \cdot \langle 1, 0 \rangle = 12 \quad (= f_x(1,2))$.

Q: What value of θ gives the max (or min) rate of change?

Ans $D_\theta f = \nabla f \cdot u_\theta = |\nabla f| \underbrace{|u_\theta|}_{=1} \cos\varphi$ → angle between ∇f and u_θ

$$= |\nabla f| \cos\varphi.$$

Well $\cos\varphi$ is a max when $\varphi = 0$ and $\cos 0 = 1$. Thus the max of $D_\theta f$ is $|\nabla f|$ when u_θ and ∇f point in the same direction. The min is $-|\nabla f|$ and occurs when $\varphi = \pi$:



Ex Let $f(x,y) = xe^y$. What is the max rate of change and in what direction is it, at $(0,0)$?

Sol $\nabla f = \langle e^y, xe^y \rangle = \langle 1, 0 \rangle$.

Thus, the max rate of change is $|\nabla f| = 1$ along the x-axis.

Here is a variation:

Ex Let $f(x,y) = x^2y + 2x$. Let $\mathbf{v} = 3i - j$
Find the rate of change of f in the direction $\mathbf{v}$ at (x,y).

Sol Let $u = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 3, -1 \rangle}{\sqrt{10}} = \langle \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \rangle$

$$D_u f = \nabla f \cdot u = \langle 2xy + 2, x^2 \rangle \cdot \langle \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \rangle$$

$$= \frac{6xy + 6 - x^2}{\sqrt{10}}.$$

All this works in 3-d.

Ex Let $v = 2i - 3j + k$. Let $f(x,y,z) = x^3 y e^z$. Find the rate of change of $f(x,y,z)$ at the point $(1,1,0)$ in the direction of v.

Sol. Let $u = \frac{v}{|v|} = \frac{\langle 2,-3,1\rangle}{\sqrt{14}} = \left\langle \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle$

Let $r(t) = ut + \langle 1,1,0\rangle$. This is a line starting a $(1,1,0)$ and moving in the direction u. Then

$$D_u f = \frac{d\, f(r(t))}{dt} = \nabla f \cdot r'(t) = \nabla f \cdot u$$

Same as in the 2-d case. Thus

$$D_u f = \nabla f \cdot u = \langle 3x^2 y e^z, x^3 e^z, x^3 y e^z \rangle \cdot u$$

$$= \langle 3,1,1\rangle \cdot \left\langle \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle = \frac{6-3+1}{\sqrt{14}} = \frac{4}{\sqrt{14}} \approx 1.069$$

Fact The direction of max rate of change of $f(x,y,z)$ is ∇f and has value $|\nabla f|$. The direction of the min rate of change is $-\nabla f$ and has value $-|\nabla f|$.

The proof is the same as the 2-d case.

Tangent Planes to Level Surfaces

Idea ∇F is normal to a level surface. Explain:
Let $F(x,y,z) = C$. Let $r(t) = \langle x(t), y(t), z(t)\rangle$ be any curve s.t. $F(r(t)) = C$. Then

$$\frac{d\,F(r(t))}{dt} = 0$$

$$\nabla F \cdot r'(t) = 0$$

So ∇F is $\perp$ to any vector in the tangent plane.

Ex Let $F(x,y,z) = 4x^2y^3z + 7z^2x + 4xyz^4$.
Consider the level surface $F(x,y,z) = 15$.
Find the tangent ~~vect~~ plane at $(1,1,1)$.

Sol $\nabla F = \langle F_x, F_y, F_z\rangle = \langle 8xy^3z + 7z^2 + 4yz^4,\ 12x^2y^2z + 0 + 4xz^4,$
$4x^2y^3 + 14zx + 16xyz^3\rangle$

At $(1,1,1)$ $\nabla F = \langle 19, 16, 34\rangle$.

Thus the tangent plane is $\langle 19,16,34\rangle \cdot \langle x-1, y-1, z-1\rangle = 0$
or

$$19x + 16y + 34z = 69.$$