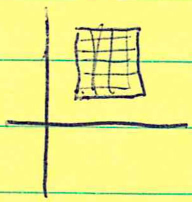
Chapter 12 Sections 12.1 & 12.2

Ex Find the volume under $f(x,y) = x + y^2$ over the rectangle $[1,3]\times[2,4]$.

Method I Use sums to approximate the volume.



Let $1 = x_0 < x_1 < x_2 < \cdots < x_m = 3$ and
$2 = y_0 < y_1 < y_2 < \cdots < y_m = 4$.
Assume they are equally spaced. Let $\Delta x = x_{i+1} - x_i = \frac{3-1}{m}$
Let $\Delta y = \frac{4-2}{n} = y_{i+1} - y_i$.

$$\text{Then Vol} \approx \sum_{j=0}^{n-1}\left(\sum_{i=0}^{m-1} f(x_i, y_j)\,\Delta x\right)\Delta y.$$

I used a computer (Maple) to do this for serval choices of m and n. The number of calculations is very large. For $n = m = 1000$, there are 10^6 passes through the summation loop. It took a couple of minutes. Here are my results

$m=n=5$	$m=40$ $n=50$	$m=n=100$	$m=n=1000$
39.84	44.7544	45.0536	45.305336

To get the exact answer we take the limit as $n, m \to \infty$, $\Delta x, \Delta y \to 0$. Then we get

$$\int_2^4 \int_1^3 f(x,y)\, dx\, dy$$

In our example

$$\int_2^4 \left(\int_1^3 x + y^2\, dx \right) dy = \int_2^4 \left[\frac{x^2}{2} + xy^2 \right]_1^3 dy$$

$$= \int_2^4 \left[\frac{9}{2} + 3y^2 \right] - \left[\frac{1}{2} + y^2 \right] dy = \int_2^4 4 + 2y^2\, dy$$

$$4y + \frac{2}{3} y^3 \Big|_2^4 = \left(16 + \frac{\overbrace{2 \cdot 64}^{128}}{3} \right) - \left(8 + \frac{16}{3} \right)$$

$$= 8 + \frac{112}{3} = \frac{24}{3} + \frac{112}{3} = \frac{136}{3} = 45\tfrac{1}{3}$$

Find the approximate volume of f(x,y) over the rectangle [a,b]x[c,d].

```
> f:= (x,y) -> x + y^2;
```

$$f := (x, y) \to x + y^2$$

```
>  a:=1: b:= 3: c:=2: d:=4: m:=5: n:=5: dx := (b-a)/m; dy := (d-c)/n;
```

$$dx := \frac{2}{5}$$

$$dy := \frac{2}{5}$$

```
> Volume:=0:
  for i from 0 to m-1 do
    for j from 0 to n-1 do Volume := Volume + f(a+i*dx,c+j*dy)*dx*dy
    end do
  end do;
  evalf(Volume);
```

$$39.84000000$$

```
> m:=40:n:=50:dx := (b-a)/m; dy := (d-c)/n;
```

$$dx := \frac{1}{20}$$

$$dy := \frac{1}{25}$$

```
> Volume:=0:
  for i from 0 to m-1 do
    for j from 0 to n-1 do Volume := Volume + f(a+i*dx,c+j*dy)*dx*dy
    end do
  end do;
  evalf(Volume);
```

$$44.75440000$$

```
> m:=100:n:=100:dx := (b-a)/m; dy := (d-c)/n;
```

$$dx := \frac{1}{50}$$

$$dy := \frac{1}{50}$$

```
> Volume:=0:
  for i from 0 to m-1 do
    for j from 0 to n-1 do Volume := Volume + f(a+i*dx,c+j*dy)*dx*dy
```

```
    end do
  end do;
  evalf(Volume);
```

45.05360000

```
> m:=1000:n:=1000:dx := (b-a)/m; dy := (d-c)/n;
```

$$dx := \frac{1}{500}$$

$$dy := \frac{1}{500}$$

```
> Volume:=0:
  for i from 0 to m-1 do
    for j from 0 to n-1 do Volume := Volume + f(a+i*dx,c+j*dy)*dx*dy
    end do
  end do;
  evalf(Volume);
```

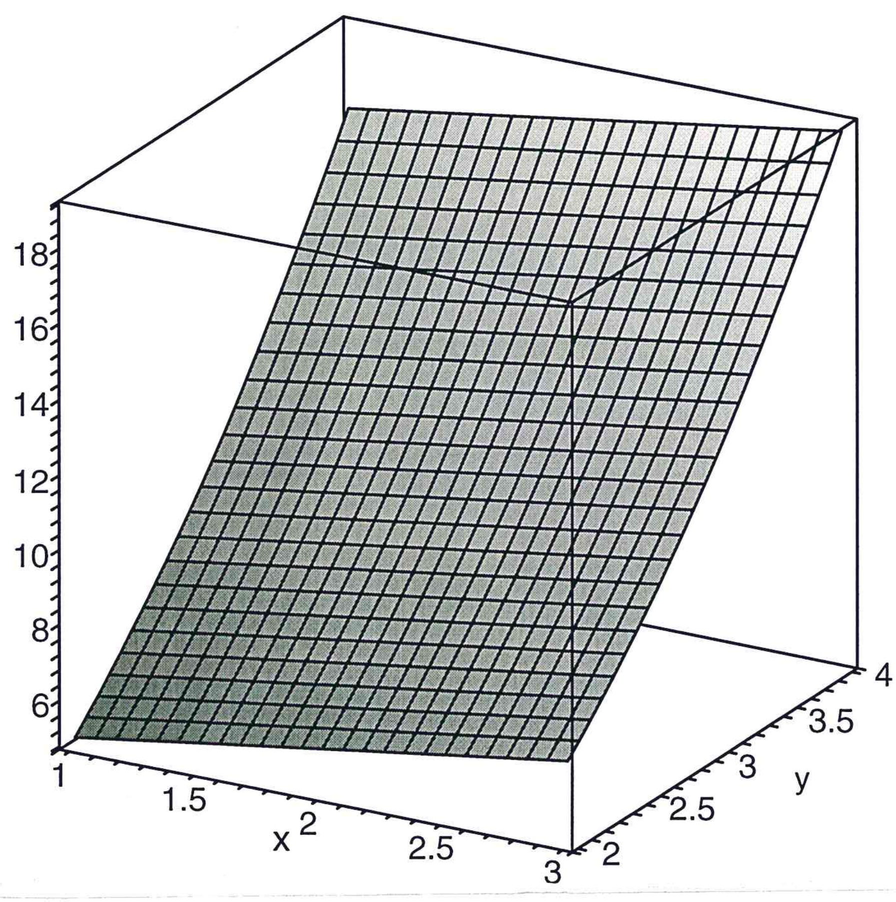
45.30533600

Now we compare to the double integral.

```
> evalf(int(int(x+y^2,x=1..3),y=2..4));
```

45.33333333

```
> plot3d(x+y^2,x= 1..3,y= 2..4);
```



Ex Find the volume under $z = x^2y + xy^2$ over $[1,2] \times [3,6]$

$$V = \int_3^6 \int_1^2 x^2y + xy^2 \, dx \, dy$$

$$= \int_3^6 \frac{x^3y}{3} + \frac{x^2y^2}{2} \Big|_1^2 \, dy$$

$$= \int_3^6 \left[\frac{8}{3}y + 2y^2\right] - \left[\frac{1}{3}y + \frac{1}{2}y^2\right] dy$$

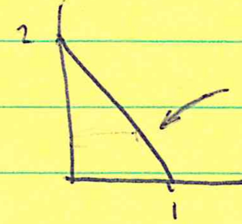
$$= \int_3^6 \frac{7}{3}y + \frac{3}{2}y^2 \, dy = \frac{7}{6}y^2 + \frac{1}{2}y^3 \Big|_3^6$$

$$= (42 + 108) - \underbrace{\left(\frac{21}{2} + \frac{27}{2}\right)}_{= \cancel{16}\ 24}$$

$$= 150 - \cancel{16} = \cancel{134}\ 126$$

$$\boxed{\text{Volume} = \cancel{134}}\ 126$$

__Ex__ Find Volume under $z = x^2 y^2$ over the triangle



$X + \frac{1}{2} Y = 1$

Let's do $dx\,dy$. Then $V = \int_0^2 \int_0^{x = -\frac{1}{2}y + 1} x^2 y^2 \, dx\, dy$

$$= \int_0^2 \frac{1}{3} x^3 y^2 \Big|_0^{-\frac{1}{2}y+1} dy = \int_0^2 \frac{1}{3}\left(1 - \frac{1}{2}y\right)^3 y^2 \, dy$$

$$= \frac{1}{3}\int_0^2 \left(1 - \frac{3}{2}y + \frac{3}{4}y^2 - \frac{1}{8}y^3\right) y^2 \, dy$$

$$= \frac{1}{3}\int_0^2 y^2 - \frac{3}{2}y^3 + \frac{3}{4}y^4 - \frac{1}{8}y^5 \, dy =$$

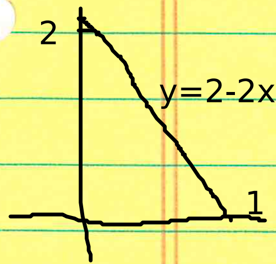
$$= \frac{1}{3}\left[\frac{1}{3}y^3 - \frac{3}{8}y^4 + \frac{3}{20}y^5 - \frac{1}{48}y^6\right]_0^2$$

$$= \frac{1}{3}\left[\frac{8}{3} - 6 + \frac{24}{5} - \frac{4}{3}\right] = \frac{1}{45}\left[40 - 90 + 72 - 20\right.$$

$$112 - 110$$

$$= \frac{2}{45} = .0\overline{4}$$

Ex   Same problem, but now do $dy\,dx$



$$\int_0^1 \int_0^{2-2x} x^2y^2 \, dy\, dx$$

$$= \int_0^1 \frac{1}{3}x^2y^3 \Big|_0^{2-2x} dx = \int_1^3 \frac{1}{3}x^2(2-2x)^3 \, dx$$

$$= \frac{8}{3}\int_0^1 x^2(1-3x+3x^2-x^3)\, dx$$

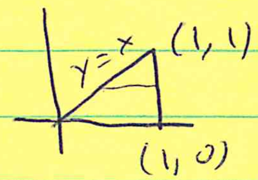
$$= \frac{8}{3}\int_0^1 x^2-3x^3+3x^4-x^5 \, dx$$

$$= \frac{8}{3}\left(\frac{1}{3}-\frac{3}{4}+\frac{3}{5}-\frac{1}{6}\right)$$

$$= \frac{8}{3}\left(\frac{-5}{12}+\frac{13}{30}\right)$$

$$= \frac{8}{3}\left(\frac{-25+26}{60}\right) = \frac{8}{3}\,\frac{1}{60} = \frac{2}{3\cdot 15} = \frac{2}{45} = .0\bar{4}$$

Ex Find the volume under $z = y e^{x^3}$ over the triangle



First we try $dx\,dy$

$$V = \int_0^1 \int_y^1 y e^{x^3}\, dx\, dy = ? \text{ stuck?}$$

Next try $dy\,dx$

$$V = \int_0^1 \int_0^x y e^{x^3}\, dy\, dx$$

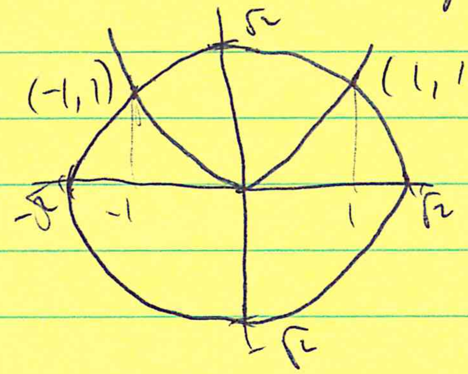
$$= \int_0^1 \tfrac{1}{2} y^2 e^{x^3} \Big|_0^x \, dx$$

$$= \int_0^1 \tfrac{1}{2} x^2 e^{x^3}\, dx \qquad u = x^3 \quad du = 3x^2 dx$$

$$= \frac{1}{6} \int_0^1 e^u\, du = \frac{1}{6} e^u \Big|_0^1 = \frac{1}{6}(e-1)$$

$$\approx 0.286380305$$

Ex Find the area of the region $R = \{(x,y) \mid x^2+y^2 \le 2 \text{ and } y \ge x^2\}$



1 Find intersection points. Sub. $y = x^2$ into $x^2+y^2=2$ to get $y^2+y-2=0$. So $y=1$ or -2. Only $y=1$ gives real solutions for x. These points are $(-1,1)$ and $(1,1)$.

2. Think of $f(x,y)=1$. Do $dy\,dx$

$$A = \int_{-1}^{1}\int_{x^2}^{\sqrt{2-x^2}} 1 \, dy\,dx$$

$$= \int_{-1}^{1} \sqrt{2-x^2} - x^2 \, dx = 2\int_0^1 \sqrt{2-x^2}\,dx - \underbrace{2\int_0^1 x^2\,dx}_{=2/3}$$

$$\int_0^1 \sqrt{2-x^2}\,dx = \int_0^{\pi/4} 2\cos^2\theta \, d\theta$$

Let $x = \sqrt{2}\sin\theta$

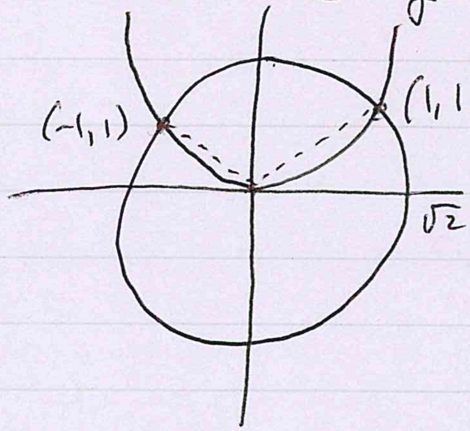
Then $dx = \sqrt{2}\cos\theta\,d\theta$

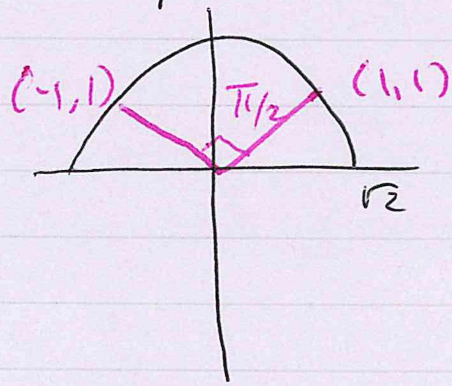
$$= \int_0^{\pi/4} \cancel{2}\,\frac{1+\cos(2\theta)}{\cancel{2}}\,d\theta$$

$$= \left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi/4} = \frac{\pi}{4} + \frac{1}{2}\cdot 1 - (0+0) = \frac{\pi}{4}+\frac{1}{2}$$

$$A = 2\left(\frac{\pi}{4}+\frac{1}{2}\right) - \frac{2}{3} = \frac{\pi}{2} + \frac{1}{3}$$

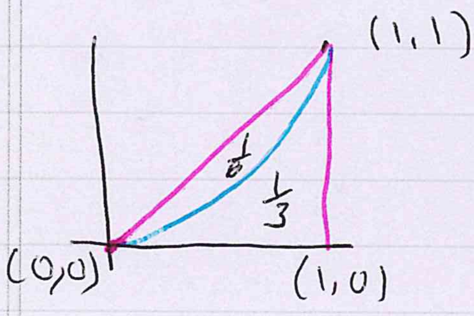
Solution using geometry.





Area of circle is $\pi r^2 = 2\pi$.
Area of one quarter sector is $\frac{1}{4}\pi r^2 = \frac{\pi}{2}$.

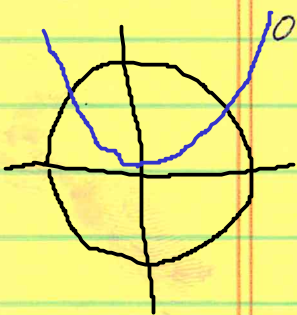
We just need to add on the two "slivers" between the dotted lines and $y = x^2$.



Area of triangle is $\frac{1}{2}$.
Area under $y = x^2$ is
$$\int_0^1 x^2\,dx = 1/3.$$
Thus, area of "sliver" is $1/6$.

Total area of region is $\frac{\pi}{2} + \frac{1}{6} + \frac{1}{6} = \boxed{\frac{\pi}{2} + \frac{1}{3}}$

Ex Find the volume under $f(x,y) = 7 + x + 2y$ over R from last time.



$$\int_{-1}^{1}\int_{x^2}^{\sqrt{2-x^2}} 7 + x + 2y \; dy\, dx$$

$$= \int_{-1}^{1} \left(7y + xy + y^2\right)\Big|_{x^2}^{\sqrt{2-x^2}} \, dx$$

$$= \int_{-1}^{1} \left[7\sqrt{2-x^2} + x\sqrt{2-x^2} + (2-x^2)\right] - \left[7x^2 + x^3 + x^4\right] d$$

$$\int_{-1}^{1} 7\sqrt{2-x^2} + \cancelto{0}{x\sqrt{2-x^2}} - x^4 - \cancelto{0}{x^3} - 8x^2 + 2 \; dx$$

$$= 7\left(\frac{\pi}{2} + 1\right) - \frac{2}{5} - 8\cdot\frac{2}{3} + 4$$

$$= \frac{7\pi}{2} + 11 - \frac{2}{5} - \frac{16}{3}$$

$$= \frac{7\pi}{2} + \frac{165}{15} - \frac{6}{15} - \frac{80}{15} = \boxed{\frac{7\pi}{2} + \frac{79}{15}}$$

$$\begin{array}{r} 165 \\ -86 \\ \hline 79 \end{array}$$

```
int(int(7+x+2*y,y=x^2..sqrt(2-x^2)),x=-1..1);
```

$$\frac{7}{2}\pi + \frac{79}{15}$$