12.3 Double Integrals in Polar Coordinates

Review

$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$

$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$, but check the quadrant

Ex Convert $r = \tan\theta\sec\theta$ to rect. coor's.

Sol

$r\cos\theta = \tan\theta.$

$x = \frac{y}{x} \quad$ or $\quad y = x^2.$

Ex Convert $r = \sin\theta$ to rect. coor's and graph.

Sol

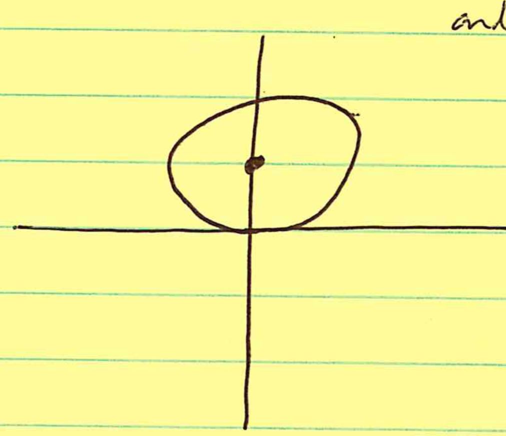
$r = \sin\theta$

$r^2 = r\sin\theta$

$x^2+y^2 = y$

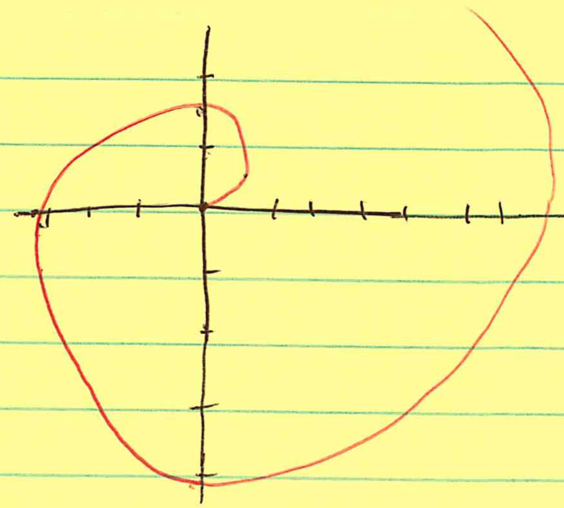
$x^2+y^2-y+\frac{1}{4} = 0+\frac{1}{4}$

$x^2+(y-\frac{1}{2})^2 = \frac{1}{4}$ circle with center $(0,\frac{1}{2})$ and radius $\frac{1}{2}$.



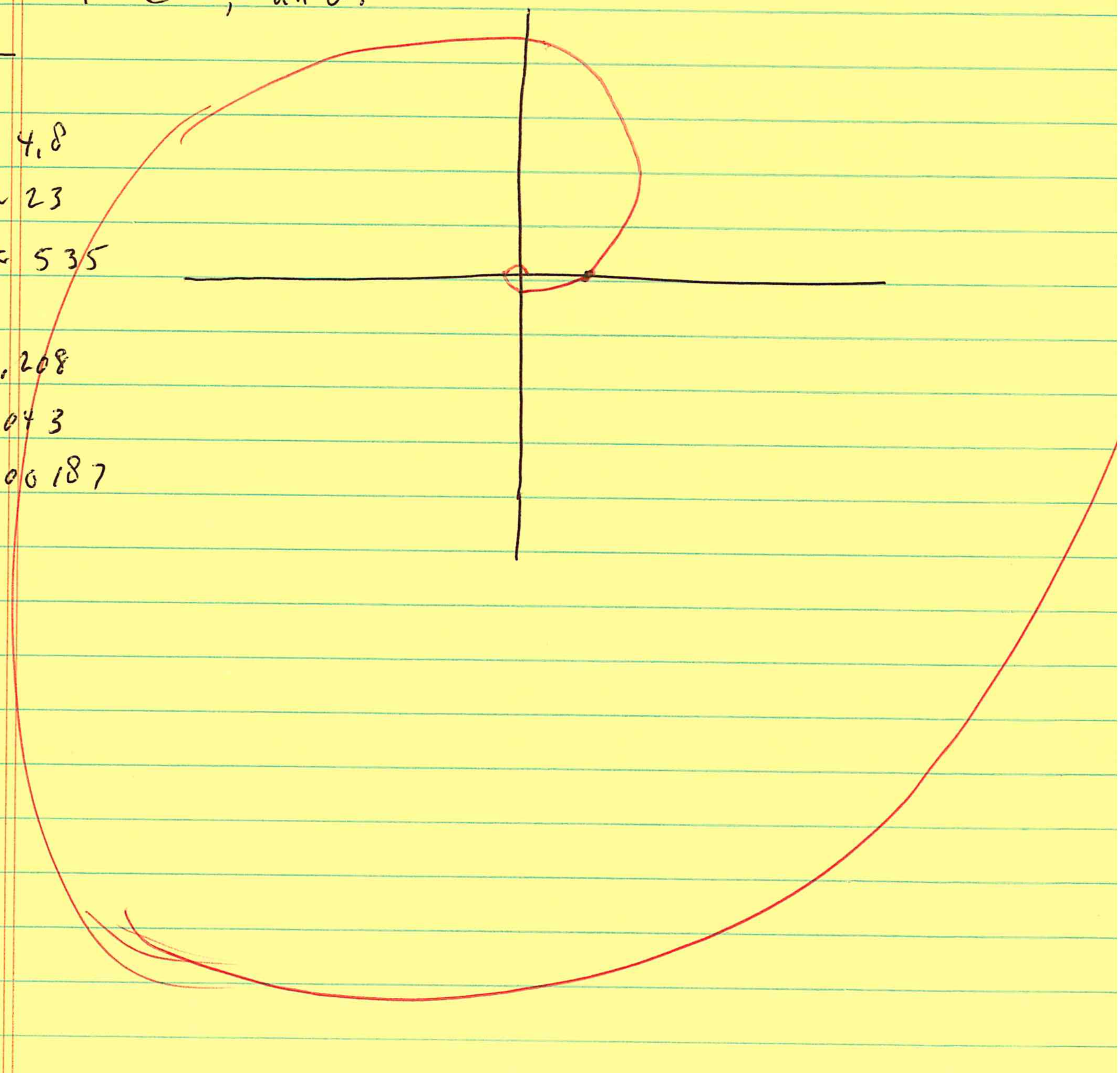
Graphs: $r=\theta$, for $\theta \geq 0$.

θ	r
0	0
$\frac{\pi}{4}$	$\frac{\pi}{4} \approx .78$
$\frac{\pi}{2}$	$\frac{\pi}{2} \approx 1.57$
π	$\pi \approx 3.14$
$\frac{3\pi}{2}$	≈ 4.7
2π	≈ 6.28



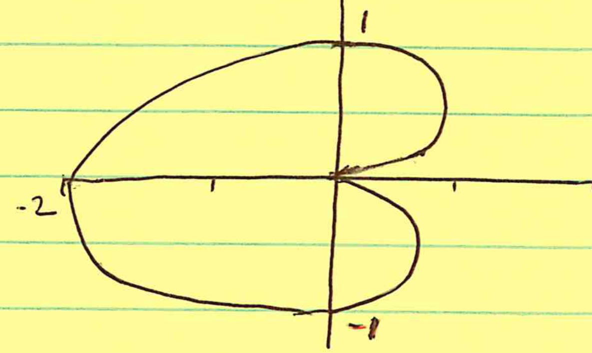
$r=e^{\theta}$, all θ.

θ	r
0	1
$\frac{\pi}{2}$	≈ 4.8
π	≈ 23
2π	≈ 535
$-\frac{\pi}{2}$	$.208$
$-\pi$	$.043$
-2π	$.00187$



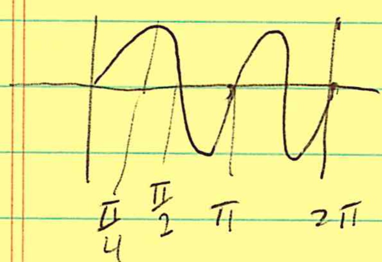
Graph $r = 1 - \cos\theta$

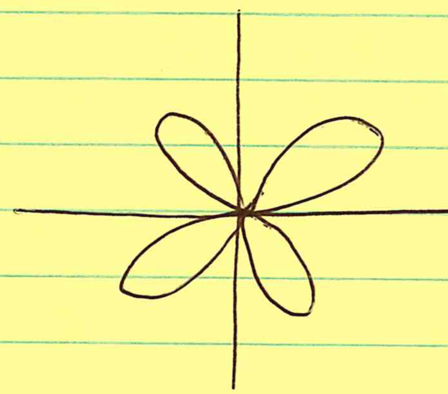
θ	r
0	0
$\frac{\pi}{4}$	$1 - \frac{\sqrt{2}}{2} \approx .293$
$\frac{\pi}{2}$	1
π	2
$3\frac{\pi}{2}$	1
2π	0



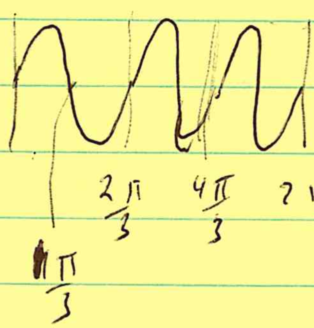
Cardioid

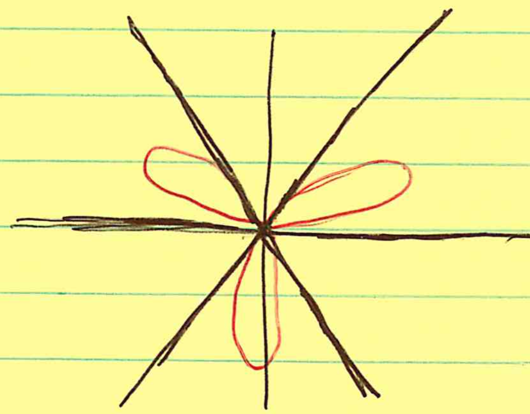
Graph $r = \sin 2\theta$





$r = \sin 3\theta$



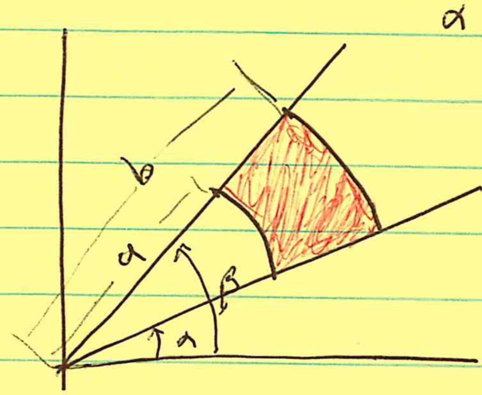


repeats

Area

A polar rectangle is a region in $\mathbb{R}^2$ determined by

$$a \le r \le b,$$
$$\alpha \le \theta \le \beta.$$



We shall derive a formula for its area.

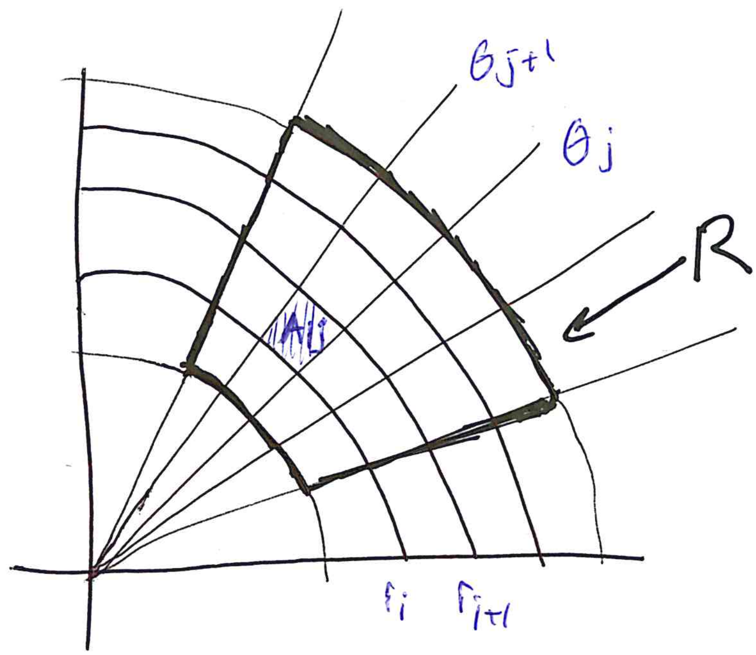
The area A is $=$ big sector $-$ small sector

$$= \pi b^2\left(\frac{\beta-\alpha}{2\pi}\right) - \pi a^2\left(\frac{\beta-\alpha}{2\pi}\right)$$

$$= \frac{1}{2}(b^2-a^2)(\beta-\alpha).$$

Volume

To do integration of some function $f(r,\theta)$ over a region R using polar coordinates we divide the region up into small polar rectangles. We set up a double Riemann Sum and make the polar rectangle become smaller and smaller. (Roughly speaking.)


θj+1
θj
R
ΔAij
ri
ri+1

If the region R is a large polar rect. $(a \le r \le b,\ \alpha \le \theta \le \beta)$
we can partition it into smaller polar rect's.

$$a = r_0 < r_1 < r_2 < \cdots < r_m = b$$
$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = \beta$$

Let A_{ij} be the area of the small polar rect

$$r_i \le r \le r_{i+1}$$
$$\theta_j \le \theta \le \theta_{j+1}.$$

Then the vol. under $f(r,\theta)$ over R is approximated by

$$\text{Vol} \approx \sum_{j=0}^{n-1} \sum_{i=\emptyset}^{m-1} f(r_i^*, \theta_j^*)\, A_{ij}$$

where (r_i^*, θ_j^*) is some point in A_{ij}. But, this is not (yet) in the form of a Riemann sum. So, we cannot take a limit to get an integral. We do the following:

$$A_{ij} = \frac{1}{2}\left(r_{i+1}^2 - r_i^2\right)\underbrace{\left(\theta_{j+1} - \theta_j\right)}_{\Delta\theta} = \frac{1}{2}\left(r_{i+1} + r_i\right)\left(r_{i+1} - r_i\right)\Delta\theta$$

$$= \frac{r_{i+1} + r_i}{2}\,\Delta r\,\Delta\theta$$

Use $r_i^* = \dfrac{r_{i+1} - r_i}{2}$. Now we have

$$V \approx \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} f(r_i^*, \theta_j^*)\, r_i^* \Delta r \Delta \theta$$

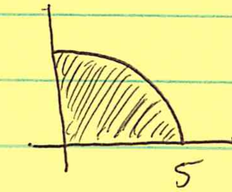
This is now a (double) Riemann Sum. As $\Delta r, \Delta\theta \to 0$ and $n, m \to \infty$ we get

$$V = \int_\alpha^\beta \int_a^b f(r,\theta) \underbrace{r\,dr\,d\theta}_{dA}$$

One way to remember why we need $r\,dr\,d\theta$ is that if we used $dr\,d\theta$ the units would be wrong!

Ex Find the area bounded by $r=5$, $\theta=0$, $\theta=\frac{\pi}{2}$.

Sol Obviously we should get $\frac{25\pi}{4}$.



$$\int_0^{\frac{\pi}{2}}\int_0^5 1\; r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \frac{r^2}{2}\Big|_0^5\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{25}{2}\,d\theta = \frac{25}{2}\left(\theta\Big|_0^{\pi/2}\right) = \frac{25\pi}{4}.$$

Ex Find the volume under the plane $2x+y+3z=30$ over the unit disk, $x^2+y^2\le 1$.

Sol Solve for z to get $z=\frac{30-2x-y}{3}$. The integral is

$$V=\int_0^{2\pi}\int_0^1 \frac{30-2r\cos\theta - r\sin\theta}{3}\; r\,dr\,d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}\int_0^1 30r - 2r^2\cos\theta - r^2\sin\theta\; dr\,d\theta$$

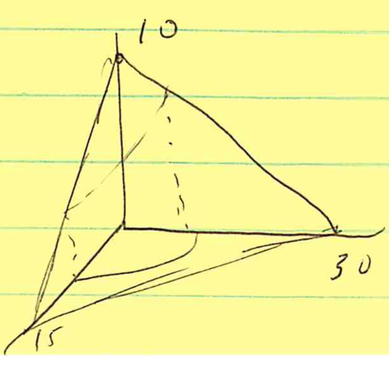
$$=\frac{1}{3}\int_0^{2\pi} \frac{30r^2}{2} - \frac{2}{3}r^3\cos\theta - \frac{r^3}{3}\sin\theta\,\Big|_{r=0}^{r=1}\; d\theta$$

$$=\frac{1}{3}\int_0^{2\pi} 15 - \frac{2}{3}\cos\theta - \frac{1}{3}\sin\theta\; d\theta$$

$$=\frac{1}{3}\cdot 15\cdot 2\pi = 10\pi.$$

(Think about this: base $\times$ ave height.)

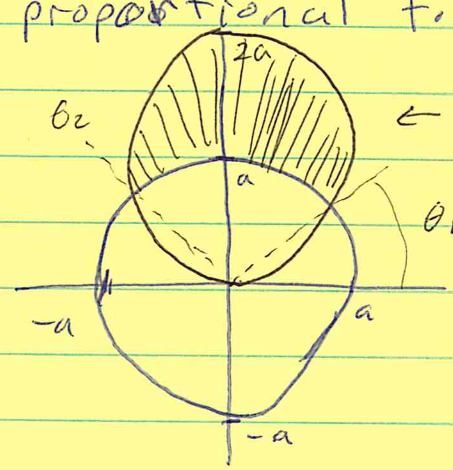
$\qquad\qquad\quad \overset{\shortparallel}{\pi} \qquad\quad \overset{\shortparallel}{10}$



Ex Let R be the region outside $r = a$ but inside $r = 2a\sin\theta$.

1. Find the area of R.
2. Find the mass of a metal plate in the shape and location of R if the density is inversely proportional to the distance from the origin.

Sol 1



← This is R.

We need the angles where the intersections occur.

$$2a\sin\theta = a$$
$$\sin\theta = \frac{1}{2}$$
$$\theta_1 = \frac{\pi}{6},\ \theta_2 = \frac{5\pi}{6}$$

$$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{a}^{2a\sin\theta} 1 \ r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{r^2}{2}\Bigg|_{a}^{2a\sin\theta} d\theta = \frac{a^2}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 4\sin^2\theta - 1 \ d\theta$$

$$\cancel{\int \sin^2\theta\,d\theta = \int \frac{1-\cos(2\theta)}{2}\,d\theta = \frac{1}{2}\theta - \frac{1}{2}\cdot\frac{1}{2}\sin(2\theta)}$$

$$\cancel{\text{Area} = \frac{a^2}{2}\left[2\theta - \sin(2\theta) - \theta\right]}$$

Use $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$.

$$\text{Area} = \frac{a^2}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 1 - 2\cos(2\theta)\, d\theta$$

$$= \frac{a^2}{2}\Big[\theta - \sin(2\theta)\Big]_{\pi/6}^{5\pi/6}$$

$$= \frac{a^2}{2}\left(\frac{4\pi}{6} - \sin\left(\frac{5\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)\right)$$

$$= \frac{a^2}{2}\left(\frac{2\pi}{3} - \left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= a^2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right).$$

Sol of 2 Let the density be given by $\rho(r,\theta) = \dfrac{k}{r}$, for some constant k. Then

$$\text{Mass} = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{a}^{2a\sin\theta} \frac{k}{r}\, r\, dr\, d\theta$$

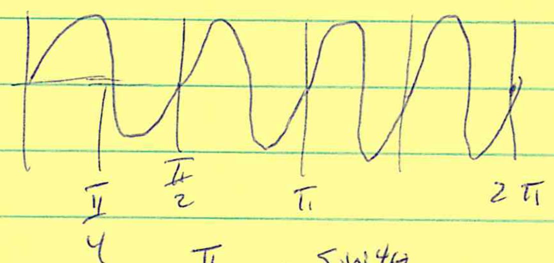
$$= k\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{a}^{2a\sin\theta} 1\, dr\, d\theta$$

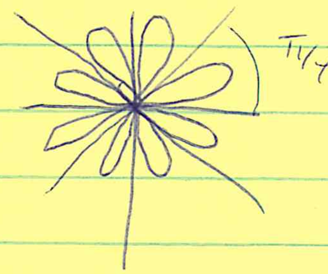
$$= k\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 2a\sin\theta - a\, d\theta = ak\Big[-2\cos\theta - \theta\Big]_{\pi/6}^{5\pi/6}$$

$$= ak\left[\left(-2\left(-\frac{\sqrt{3}}{2}\right) - \frac{5\pi}{6}\right) - \left(-2\left(\frac{\sqrt{3}}{2}\right) - \frac{\pi}{6}\right)\right]$$

$$= 9k\left(2\sqrt{3} - \frac{2\pi}{3}\right).$$

Ex Let $r = \sin 4\theta$. Find the area of one petal.





$$\text{Area} = \int_0^{\frac{\pi}{4}} \int_0^{\sin 4\theta} 1 \; r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \frac{\sin^2 4\theta}{2}\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1-\cos(8\theta)}{2}\,d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{1}{8}\sin(8\theta)\right)\Big|_0^{\pi/4}$$

$$= \frac{1}{4}\left(\left[\frac{\pi}{4} - \frac{1}{8}\underbrace{\sin(2\pi)}_{=0}\right] - \left[0 - 0\right]\right)$$

$$= \frac{\pi}{16}.$$