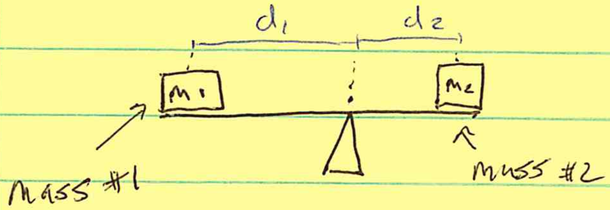
12.4 Applications

Note: Students should review Section 7.6

Center of Mass

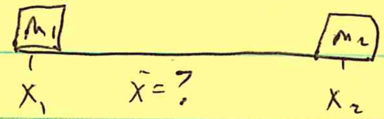


balenced $\iff m_1 d_1 = m_2 d_2$.

Law of the Lever

Archimedes ~ 200 B.C.

Question: Suppose we have



. How do we find the balance point $\bar{x}$?

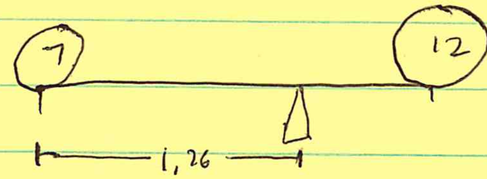
Solution Let $d_1 = \bar{x} - x_1$, $d_2 = x_2 - \bar{x}$.

Them $m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}) \Rightarrow \bar{X} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$.

Ex Given a thin strong board of length 2 meters with a 7 kg mass on one end and a 12 kg mass on the other, where is the balance pt?

Sol Let $m_1 = 7$, $x_1 = 0$, $m_2 = 12$, $x_2 = 2$. Then

$$\bar{x} = \frac{7 \cdot 0 + 12 \cdot 2}{7 + 12} = \frac{24}{19} \approx 1.26 \text{ meters.}$$



Try this at home!

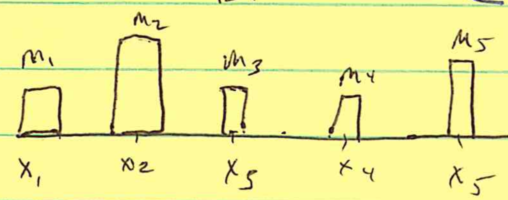
If there a several weights then it can be shown that

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}$$

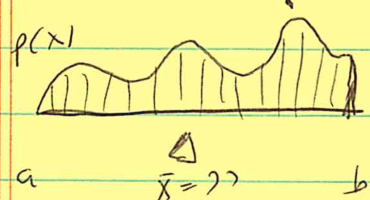
$$= \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

This is called a weighted average.

Total mass.



Now let $\rho(x)$ be a linear density function : mass/length



To find the balance point partition $[a,b]$ by $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Let $m_i = \rho(x_i)\Delta x$.

Then

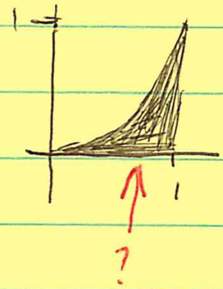
$$\bar{x} \approx \frac{\sum_{i=0}^{n-1} m_i x_i}{\sum_{i=1}^{n-1} m_i} = \frac{\sum_{i=0} \rho(x_i)\, x_i \,\Delta x}{\sum_{i=0}^{n-1} \rho(x_i)\,\Delta x}.$$

Now, take limit as $\Delta x \to 0$, $n \to \infty$, to get

$$\bar{x} = \frac{\int_a^b \rho(x)\, x \, dx}{\int_a^b \rho(x)\, dx}.$$

Ex Let $\rho(x) = x^2$ over $[0,1]$. Find the balance pt. $\bar{x}$.

Sol

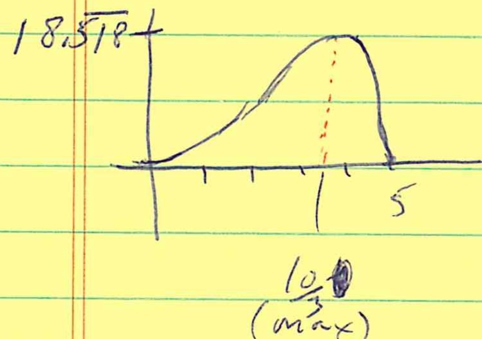


$$\bar{x} = \frac{\int_0^1 x^2 \cdot x\,dx}{\int_0^1 x^2\,dx} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}.$$

Thus $\bar{x} = \frac{3}{4}$.

Ex Let $\rho(x) = x^2(5-x)$ over $[0,5]$. Find $\bar{x}$.

Compare to location of max of $p(x)$.



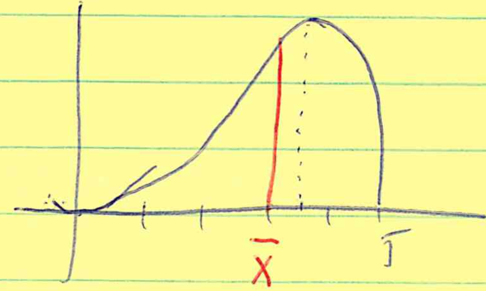
Sol To find max: $p'(x) = (5x^2 - x^3)' = 10x - 3x^2$

$$= x(10-3x) \overset{\text{set}}{=} 0$$

$$\Rightarrow x = \frac{10}{3}$$

Max is at $x = \frac{10}{3} = 3\frac{1}{3}$. $f(3\frac{1}{3}) = \frac{500}{27} = 18.\overline{518}$.

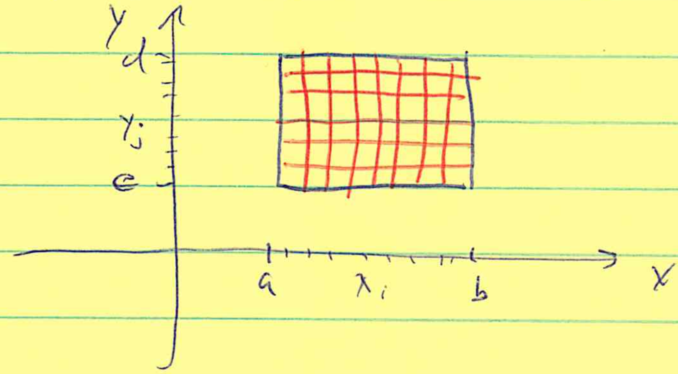
To find $\bar{x}$: $\bar{x} = \dfrac{\int_0^5 5x^3 - x^4\,dx}{\int_0^5 5x^2 - x^3\,dx} = \cdots = 3.$



Now we generalize to two dimensions.

Let $\rho(x,y)$ be a density function $\left(\frac{\text{mass}}{\text{area}}\right)$ for a rectangle $R = [a,b] \times [c,d]$. We want to find the balance point $(\bar{x}, \bar{y})$, also called the <u>center of mass</u>.

We partition R as shown



$$a = x_0 < x_1 < x_2 < \cdots < x_m = b$$
$$c = y_0 < y_1 < y_2 < \cdots < y_n = d$$
$$\Delta x = x_{i+1} - x_i$$
$$\Delta y = y_{j+1} - y_j.$$

Let $m_{ij} = \rho(x_i, y_j)\Delta x \Delta y \approx$ mass of cell ij.

Let $M_i = \sum_{j=0}^{m-1} m_{ij} \approx$ mass of column i

Then

$$\bar{x} \approx \frac{\sum_{i=0}^{m-1} M_i x_i}{\sum_{i=0}^{m-1} M_i} = \frac{\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} x_i \rho(x_i,y_j)\Delta x \Delta y}{\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \rho(x_i,y_j)\Delta x \Delta y}$$

Taking limits as $\Delta x, \Delta y \to 0$ $(m, n \to \infty)$ we get

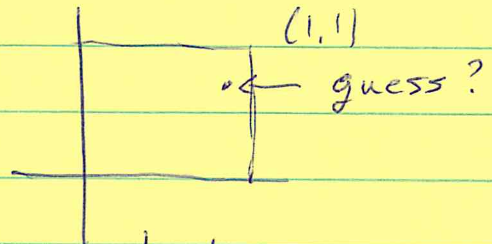
$$\bar{x} = \frac{\int_c^d \int_a^b x\,\rho(x,y)\,dx\,dy}{\int_c^d \int_a^b \rho(x,y)\,dx\,dy} = \frac{M_y}{M} \begin{matrix} \leftarrow \text{moment wrt } y\text{-axis} \\ \leftarrow \text{total mass} \end{matrix}$$

Likewise $\bar{y} = \dfrac{M_x}{M} = \dfrac{\iint y\,\rho(x,y)\,dx\,dy}{\iint \rho(x,y)\,dx\,dy}$.

This holds for regions that are not rectangles.

Ex Find the mass and center of mass of the square $[0,1]\times[0,1]$ with density function $\rho(x,y) = x^2+2y+1$

Sol



$$M = \int_0^1\int_0^1 x^2+2y+1\,dx\,dy = \int_0^1 \tfrac{1}{3}+2y+1\,dy = \tfrac{4}{3}+1 = \boxed{\tfrac{7}{3}}$$

$$M_y = \int_0^1\int_0^1 x(x^2+2y+1)\,dx\,dy = \int_0^1\int_0^1 x^3+2xy+x\,dx\,dy$$

$$= \int_0^1 \tfrac{1}{4}+y+\tfrac{1}{2}\,dy = \tfrac{3}{4}+\tfrac{1}{2} = \boxed{\tfrac{5}{4}}$$

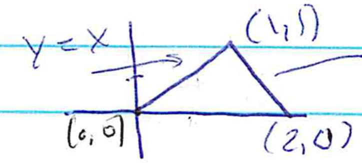
$$M_x = \int_0^1\int_0^1 y(x^2+2y+1)\,dx\,dy = \int_0^1\int_0^1 x^2y+2y^2+y\,dx\,dy$$

$$= \int_0^1 \tfrac{1}{3}y+2y^2+y\,dx = \tfrac{4}{3}\cdot\tfrac{1}{2}+\tfrac{2}{3} = \boxed{\tfrac{4}{3}}$$

Thus,

$$\boxed{\begin{aligned} \bar{x} &= \frac{M_y}{M} = \frac{5/4}{7/3} = \frac{15}{28} \approx 0.5357142 \\ \bar{y} &= \frac{M_x}{M} = \frac{4/3}{7/3} = \frac{4}{7} = \frac{16}{28} \approx 0.57142857 \end{aligned}}$$

Ex Let R be the triangular region below. Let $\rho(x,y) = xy$ be a density function. Find the mass and center of mass.



$y = -x+2$

$x = -y+2 = 2-y$

$$\boxed{\text{Mass}} = \int_0^1 \int_y^{2-y} xy \, dx\,dy = \int_0^1 \frac{1}{2} x^2 y \Big|_y^{2-y} dy$$

$$\frac{1}{2}\int_0^1 (2-y)^2 y - y^3 \, dy = \frac{1}{2}\int_0^1 4y - 4y^2 + \cancel{y^3} - \cancel{y^3} \, dy$$

$$= 2\int_0^1 y - y^2 \, dy = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{2}{6} = \boxed{\frac{1}{3}}$$

$$\boxed{M_x} = \int_0^1 \int_y^{2-y} xy^2 \, dx\,dy = \cdots = 2\int_0^1 y^2 - y^3 \, dy = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{2}{12} = \boxed{\frac{1}{6}}$$

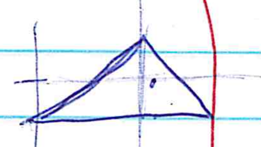
$$\boxed{M_y} = \int_0^1 \int_y^{2-y} x^2 y \, dx\,dy = \int_0^1 \frac{1}{3} x^3 y \Big|_y^{2-y} dy =$$

$$\frac{1}{3}\int_0^1 (2-y)^3 y - y^4 \, dy = \frac{1}{3}\int_0^1 [8 - 12y + 6y^2 - y^3]y - y^4 \, dy$$

$$= \frac{1}{3}\int_0^1 8y - 12y^2 + 6y^3 - 2y^4 \, dy = \frac{1}{3}\left(\frac{8}{2} - \frac{12}{3} + \frac{6}{4} - \frac{2}{5}\right)$$

$$= \frac{1}{3}\left(\frac{3}{2} - \frac{2}{5}\right) = \frac{1}{3}\left(\frac{15-4}{10}\right) = \boxed{\frac{11}{30}}$$

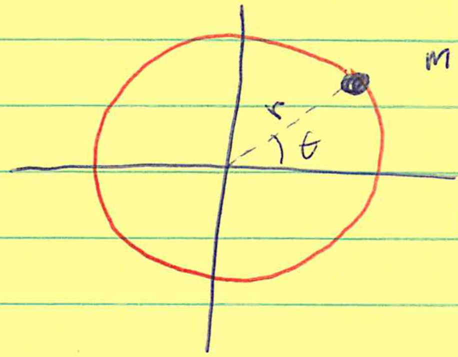
$$\bar{x} = \frac{M_y}{M} = \frac{\frac{11}{30}}{\frac{1}{3}} = \frac{11}{10} = 1.1$$



$$\bar{y} = \frac{M_x}{M} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2} = 0.5$$

Rotational Inertia

Idea



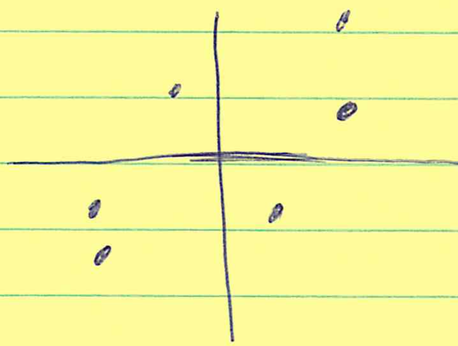
Angular velocity : $\omega = \frac{d\theta}{dt}$

Linear Velocity : $v = r\omega$

Kenetic Energy $E = \frac{1}{2}mv^2$

$$= \frac{1}{2}\underbrace{mr^2}_{I_0 \text{ (or } I_z)}\omega^2 .$$

I_0 is called the rotational inertia.



Suppose we have several objects rotating with the same ω about $(0,0)$.

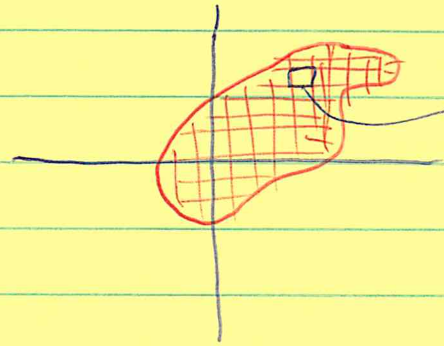
Now $E = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i r_i^2 \omega_i^2$

$$= \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

$$= \frac{1}{2} I_0 \omega^2$$

I_0 is the total rotational inertia

Now suppose we have an object is the xy-plane with density $\rho(x,y)$ and we want to find its rotational interia wrt $(0,0)$.



Let $m_{ij} = \rho(x_i, y_j)\Delta x \Delta y$

Then $E \approx \frac{1}{2}\left(\sum\sum m_{ij} r_{ij}^2\right)\omega^2$

So in limit as $\Delta x, \Delta y \to 0$ we get

$$E = \frac{1}{2}\left(\int\int r^2 \rho(x,x)\, dx\, dy\right)\omega^2.$$

We defined

$$I_0 = \iint_R r^2 \rho\, dA$$

could be

$$I_0 = \int\int (x^2+y^2)\,\rho(x,y)\, dx\, dy$$

or

$$I_0 = \int\int r^3 \rho(r,\theta)\, dr\, d\theta$$

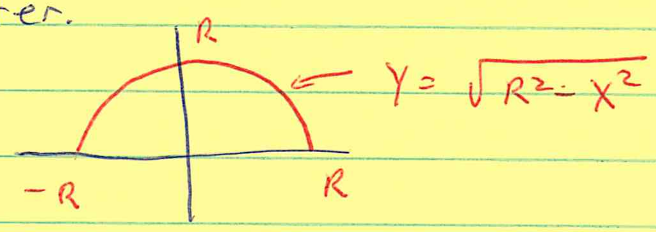
It is ~~[illegible]~~ convenient to ~~[illegible]~~ also define

$$I_x = \iint x^2 \rho(x,y)\, dx\, dy$$

$$I_y = \iint y^2 \rho(x,y)\, dx\, dy$$

Then $I_0 = I_x + I_y$.

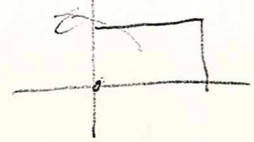
Ex Find I_0 for the semi-circle with radius R and density proportional to the distance from the diameter, where I_0 means wrt to the circle's center.



Sol Let $\rho(x,y) = ky$, for some $k > 0$.

$$I_0 = \int_0^{\pi} \int_0^{R} k r \sin\theta \, r^3 \, dr\, d\theta$$

$$= k \frac{R^5}{5} \int_0^{\pi} \sin\theta \, d\theta = \frac{2kR^5}{5}$$



Ex Find I_0 for the rectangle $[0,2]\times[0,1]$ with $\rho(x,y) = 3x+2y+1$.

Sol:

$$I_x = \int_0^1 \int_0^2 y^2(3x+2y+1)\,dx\,dy$$

$$= \int_0^1 \int_0^2 3xy^2 + 2y^3 + y^2 \,dx\,dy$$

$$\int_0^1 \frac{3x^2y^2}{2} + (2y^3+y^2)x \Big|_0^2 \,dy$$

$$= \int_0^1 6y^2 + 4y^3 + 2y^2 \,dy$$

$$= \frac{8y^3}{3} + y^4 \Big|_0^1 = \frac{8}{3} + 1 = \frac{11}{3} = 3\frac{2}{3}$$

$$I_y = \int_0^1 \int_0^2 x^2(3x+2y+1)\,dx\,dy$$

$$= \int_0^1 \int_0^2 3x^3 + 2x^2y + x^2 \,dx\,dy =$$

$$= \int_0^1 \frac{3\cdot 16}{4} + \frac{2\cdot 8y}{3} + \frac{8}{3}\,dy = \int_0^1 \frac{16}{3}y + \frac{44}{3}\,dy$$

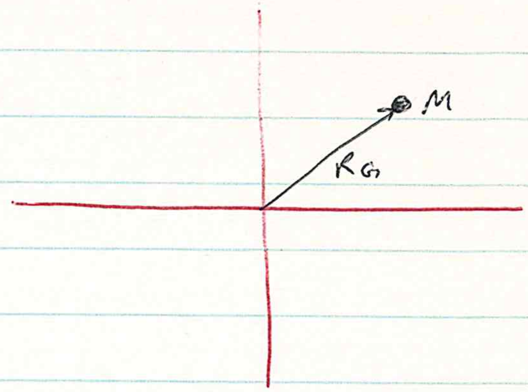
$$= \frac{8}{3} + \frac{44}{3} = \frac{52}{3} = 17\frac{1}{3}$$

$$I_0 = 3\frac{2}{3} + 17\frac{1}{3} = 21.$$

Note If the rect. is rotating about $(0,0)$ with angular vel. ω then $E = \frac{1}{2} I_0 \omega^2$.

Radius of Gyration.

Let $R_G = \sqrt{\frac{I_0}{M}}$



$I = M R_G^2 = I_0$

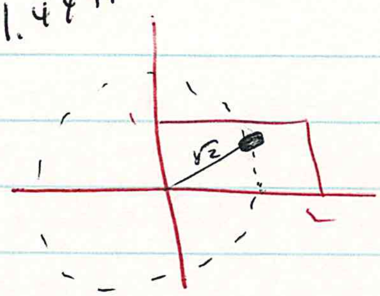
Ex The radius of Gy of the last example is,

$$R_G = \sqrt{\frac{2\cancel{0}}{M}}$$

$$M = \int_0^1 \int_0^2 3x + 2y + 1 \, dx\, dy =$$

$$= \int_0^1 6 + 4y + 2 \, dy = 8 + 2 = 10$$

$$R_G = \sqrt{\frac{21}{10}} \approx 1.449137675\ldots$$



$\cancel{E = \frac{1}{2} \cdot 10 \cdot (\sqrt{2})^2 \omega = 10\omega}$

Ex A disk with radius 1 meter and mass 3kg is moving 10 m/s in a straight line and spinning 5 revolutions per second. It has uniform density. Find the total kinetic energy.

Sol $E = \frac{1}{2}mv^2 + \frac{1}{2}I_0\omega^2$. We know m and v, and $\omega = 5\frac{\text{rev}}{\text{sec}} =$
$5\frac{\text{rev}}{\text{sec}} \cdot 2\pi \frac{\text{radians}}{\text{rev}} = 10\pi \frac{\text{radians}}{\text{sec}}$. We just need to find I_0.

$$I_0 = \int_0^{2\pi}\int_0^1 r^2 \rho \, r \, dr \, d\theta. \qquad \rho = \frac{\text{mass}}{\text{area}} = \frac{3 \text{ kg}}{\pi \text{ m}^2}$$

$$I_0 = \frac{3}{\pi}\int_0^{2\pi}\int_0^1 r^3 \, dr \, d\theta = \frac{3}{\pi}\,\frac{1}{4}\cdot 2\pi = \frac{3}{2}.$$

Thus, $E = \frac{1}{2}\cdot 3(10)^2 + \frac{1}{2}\left(\frac{3}{2}\right)(10\pi)^2 = 150 + 75\pi^2$

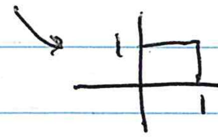
≈ 840.22 J.

Joules

$= \frac{\text{kg m}^2}{\text{s}^2}$

Additional Examples if Time Permits

Ex Let R be the square $[0,1]\times[0,1]$, Let $\rho(x,y) = xy+1$. Find the mass, the center of mass, the rotational inertia wrt to $(0,0)$ and the radius of gyration.



Sol

$$M = \int_0^1\int_0^1 xy+1\,dx\,dy = \int_0^1 \tfrac{1}{2}y+1\,dy = \tfrac{1}{4}+1 = \tfrac{5}{4}.$$

$$M_x = \int_0^1\int_0^1 y(xy+1)\,dx\,dy = \int_0^1\int_0^1 xy^2+y\,dx\,dy = \int_0^1 \tfrac{1}{2}y^2+y\,dy$$

$$= \tfrac{1}{6}+\tfrac{1}{2} = \boxed{\tfrac{2}{3}}.$$

$M_y = M_x$ by symmetry. Look for short cuts like this!

Thus $\bar{x} = \bar{y} = \dfrac{2/3}{5/4} = \dfrac{2}{3}\cdot\dfrac{4}{5} = \dfrac{8}{15}$. The $\boxed{C.M. = \left(\tfrac{8}{15},\tfrac{8}{15}\right)}$.

center of mass.

$$I_x = \int_0^1\int_0^1 y^2(xy+1)\,dx\,dy = \int_0^1\int_0^1 xy^3+y^2\,dx\,dy$$

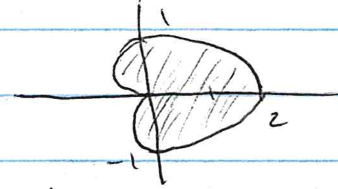
$$= \int_0^1 \tfrac{1}{2}y^3+y^2\,dy = \tfrac{1}{2}\cdot\tfrac{1}{4}+\tfrac{1}{3} = \tfrac{11}{24}.$$

$I_y = I_x$ by symmetry. Thus, $I_0 = \frac{11}{24}+\frac{11}{24} = \frac{22}{24} = \frac{11}{12}$.

$$R_G = \sqrt{\frac{I_0}{M}} = \sqrt{\frac{11/12}{5/4}} = \sqrt{\frac{11}{15}} \approx \boxed{0.8563.}$$

Done!

Ex Let R be the region determined by $0 \le r \le 1+\cos\theta$. The density is proportional to the distance from the x-axis. Find M, $\bar{x}$, $\bar{y}$, I_0, and R_G.

Sol Graph the region.  It is a cardiod — a hart shape.

Find the density function. $\rho(x,y) = k|y| = kr|\sin\theta|$.
↑ for some k

Notice both the region and the densisty are symmetric across the x-axis.

$$M = \int_0^{2\pi}\int_0^{1+\cos\theta} kr|\sin\theta| r\,dr\,d\theta = k\int_0^{2\pi}\int_0^{1+\cos\theta} r^2|\sin\theta|\,dr\,d\theta$$

$$= 2k\int_0^{\pi}\int_0^{1+\cos(\theta)} r^2\sin\theta\,dr\,d\theta = 2k\int_0^{\pi} \frac{r^3}{3}\sin\theta\Big|_0^{1+\cos\theta}\,d\theta$$

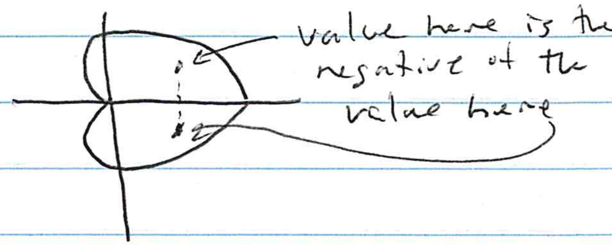
$$= \frac{2k}{3}\int_0^{\pi}(1+\cos\theta)^3\sin\theta\,d\theta.$$

Let $u = 1+\cos\theta$. Then $du = -\sin\theta\,d\theta$ and u goes from 2 to 0. Thus,

$$M = \frac{-2k}{3}\int_2^0 u^3\,du = \frac{2k}{3}\int_0^2 u^3\,du = \frac{2k}{3}\frac{16}{4} = \boxed{\frac{8}{3}k}.$$

$$M_x = \int_0^{2\pi} \int_0^{1+\cos\theta} \underbrace{(r\sin\theta)}_{= y} \, k r |\sin\theta| \, r \, dr \, d\theta = \boxed{0.}$$

Why? This integrand is odd w.r.t. y.



$$M_y = \int_0^{2\pi} \int_0^{1+\cos\theta} \underbrace{(r\cos\theta)}_{= x} \, k r |\sin\theta| \, r \, dr \, d\theta$$

$$= 2k \int_0^{\pi} \int_0^{1+\cos\theta} r^3 \cos\theta \sin\theta \, dr \, d\theta$$

$$= \frac{2k}{4} \int_0^{\pi} (1+\cos\theta)^4 \cos\theta \sin\theta \, du$$

Let $u = 1+\cos\theta$. Then $du = -\sin\theta \, d\theta$ and u goes from 2 to 0. Also notice $\cos\theta = u-1$. Thus

$$M_y = -\frac{k}{2} \int_2^0 u^4(u-1)\, du = \frac{k}{2} \int_0^2 u^5 - u \, du$$

$$\frac{16}{3} - \frac{3}{3}$$

$$= \frac{k}{2}\left(\frac{2^6}{6} - \frac{2^2}{2}\right) = k\left(\frac{2^4}{3} - 1\right) = k\left(5\tfrac{1}{3} - 1\right)$$

$$\boxed{= \frac{13}{3} k.}$$

~~C.M. is (0,~~

$$\bar{x} = \frac{0}{M} = 0. \quad \bar{y} = \frac{M_x}{M} = \frac{\frac{13}{3}k}{\frac{8}{3}k} = \frac{13}{8}$$

$$C.M. = \boxed{\left(0, \frac{13}{8}\right).}$$

We knew by symmetry.

$$I_0 = k\int_0^{2\pi}\int_0^{1+\cos\theta} r^2 \, r|\sin\theta| \, r\,dr\,d\theta =$$

$$2k\int_0^{\pi}\int_0^{1+\cos\theta} r^4 \sin\theta \, dr\,d\theta =$$

$$\frac{2k}{5}\int_0^{\pi} (1+\cos\theta)^5 \sin\theta \, d\theta.$$

Let $u = 1+\cos\theta$. Then $du = -\sin\theta \, d\theta$, $u = 2$ to 0.

$$I_0 = -\frac{2k}{5}\int_2^0 u^5\,du = \frac{2}{5}k\int_0^2 u^5\,du = \frac{2}{k}\,\frac{u^6}{6}\Big|_0^2 = \boxed{\frac{64}{15}k.}$$

$$R_G = \sqrt{\frac{I_0}{M}} = \sqrt{\frac{\frac{64}{15}k}{\frac{8}{3}k}} = \sqrt{\frac{16}{5}} = \frac{4}{\sqrt{5}} \approx \boxed{1.78885}$$

Done!