

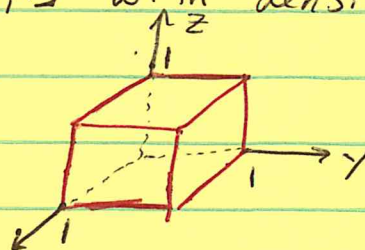
12.5

Triple Integrals

Ex Find the mass of the cube $[0, 1]^3$ with density function $\rho(x, y, z) = x^2 y z$.

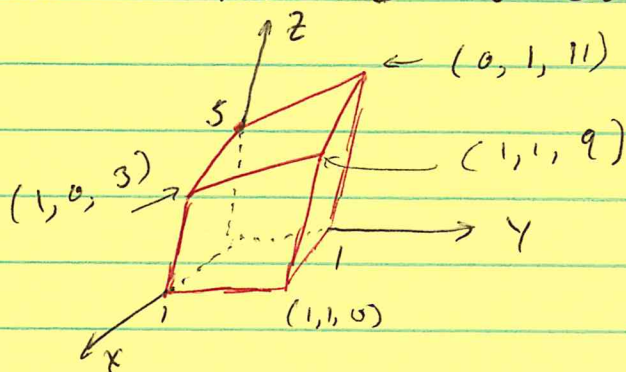
Sol Mass = $\int_0^1 \int_0^1 \int_0^1 x^2 y z \, dx dy dz$

$$= \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{12}}$$



Ex Find the mass of the region under the plane given by $2x - 6y + z = 5$ and over the square $[0, 1] \times [0, 1]$ in the xy -plane with density given by $\rho(x, y, z) = e^z$.

Sol For the plane we have $z = 5 - 2x + 6y$.



$$\text{Mass} = \int_0^1 \int_0^1 \int_0^{5-2x+6y} e^z \, dz \, dx \, dy$$

$$= \int_0^1 \int_0^1 e^{5-2x+6y} - e^0 \, dx \, dy$$

$$= \int_0^1 \int_0^1 e^{5+6y} e^{-2x} - 1 \, dx \, dy$$

$$= \int_0^1 \left(-\frac{1}{2} e^{5+6y} e^{-2x} - x \right) \Big|_0^1 \, dy$$

$$= \int_0^1 \left(-\frac{1}{2} e^{3+6y} - 1 \right) - \left(-\frac{1}{2} e^{5+6y} - 0 \right) \, dy$$

$$= \int_0^1 \frac{1}{2} (e^5 - e^3) e^{6y} - 1 \, dy$$

$$= \frac{1}{12} (e^5 - e^3) e^{6y} - y \Big|_0^1$$

$$= \frac{1}{12} (e^5 - e^3) e^6 - 1 - \frac{1}{12} (e^5 - e^3)$$

$$= \frac{(e^5 - e^3)(e^6 - 1)}{12} - 1 \approx 4302.560847$$

The next page shows how to use 3 different computer programs to do this integral.

Maple

```
> int(int(int(exp(z), z=0..5-2*x+6*y), x=0..1), y=0..1);  
- 1/12 e^5 + 1/12 e^3 + 1/12 e^11 - 1 - 1/12 e^9 (1)  
> evalf(%);  
4302.560847 (2)  
>
```

wxMaxima document

```
(%i2) integrate(integrate(integrate(exp(z), z, 0, 5-2*x+6*y), x, 0, 1), y, 0, 1);
```

```
(%o2) 
$$\frac{\%e^{11} - \%e^9 - 12}{12} - \frac{\%e^5 - \%e^3}{12}$$

```

```
(%i3) float(%);
```

```
(%o3) 4302.560847120253
```



triple integral calculator

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Computational Inputs:

- function to integrate: exp(z)
- innermost variable: z
- innermost lower limit: 0
- innermost upper limit: 5-2*x+6*y
- middle variable: x
- middle lower limit: 0
- middle upper limit: 1
- outermost variable: y
- outermost lower limit: 0
- outermost upper limit: 1

Compute

Definite integral:

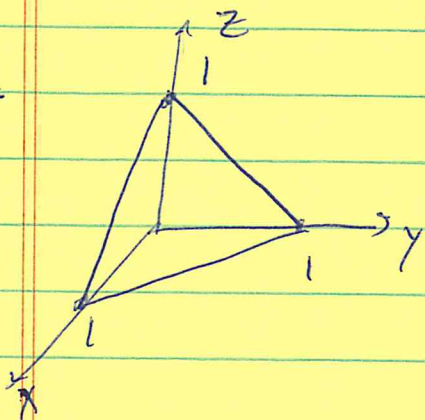
$$\int_0^1 \int_0^1 \int_0^{5-2x+6y} \exp(z) dz dx dy = \frac{1}{12} (-12 + e^3 - e^5 - e^9 + e^{11}) \approx 4302.56$$

More digits

Open code

Ex Let $\rho(x, y, z) = x$ be the density of the region in the first octant under the plane given by $x + y + z = 1$.

Sol.



You can set up the integral six different ways.

$dx dy dz$
 $dx dz dy$
 $dy dx dz$
 $dy dz dx$
 $dz dx dy$
 $dz dy dx$

I'll do $dx dy dz$ here and two others on the next page. You should do the other three

$dx dy dz$

$$\begin{aligned}
 M &= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} x \, dx \, dy \, dz \\
 &= \int_0^1 \int_0^{1-z} \frac{1}{2} (1-y-z)^2 \, dy \, dz
 \end{aligned}$$

Let $u = 1 - y - z$. Then $du = -dy$. And u goes from $1 - z$ to 0 . Thus,

$$M = -\frac{1}{2} \int_0^1 \int_{1-z}^0 u^2 \, du \, dz = -\frac{1}{2} \int_0^1 \left. \frac{u^3}{3} \right|_{1-z}^0 \, dz$$

$$= \frac{1}{6} \int_0^1 (1-z)^3 \, dz$$

Let $u = 1 - z$. Then $du = -dz$. And u goes from 1 to 0 .

$$M = -\frac{1}{6} \int_1^0 u^3 \, du = \frac{1}{6} \int_0^1 u^3 \, du = \frac{1}{6} \cdot \frac{1}{4} = \boxed{\frac{1}{24}}$$

dy dz dx

$$M = \int_0^1 \int_0^{1-x} \int_0^{1-x-z} x \, dy \, dz \, dx$$

$$= \int_0^1 \int_0^{1-x} xy \Big|_{y=0}^{y=1-x-z} dz \, dx$$

$$= \int_0^1 \int_0^{1-x} x - x^2 - xz \, dz \, dx$$

$$= \int_0^1 x(1-x) - x^2(1-x) - \frac{1}{2}x(1-x)^2 \, dx$$

$$= \int_0^1 \frac{1}{2}x^3 - x^2 + \frac{1}{2}x \, dx$$

$$= \frac{1}{8} - \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8} - \frac{1}{3} = \boxed{\frac{1}{24}}$$

dz dx dy

$$M = \int_0^1 \int_0^{1-y} \int_0^{1-x-y} x \, dz \, dx \, dy$$

$$= \int_0^1 \int_0^{1-y} x - x^2 + xy \, dx$$

$$= \int_0^1 \frac{1}{2}(1-y)^2 - \frac{1}{3}(1-y)^3 - \frac{1}{2}(1-y)^2 y \, dy$$

$$= \int_0^1 \frac{1-2y+y^2}{2} - \frac{1-3y+3y^2-y^3}{3} - \frac{y-2y^2+y^3}{2} \, dy$$

$$= \int_0^1 -\frac{1}{6}y^3 + \frac{1}{2}y^2 - \frac{1}{2}y \, dy$$

$$= -\frac{1}{24} + \frac{1}{3} - \frac{1}{4} = \frac{-1}{24} + \frac{1}{12} = \boxed{\frac{1}{24}}$$

Ex (From Swokowski, page 678).

Find the volume in \mathbb{R}^3 of the region bounded by

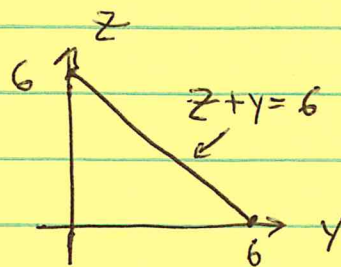
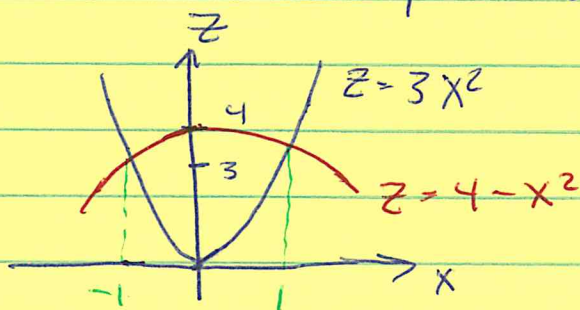
$$z = 3x^2$$

$$z = 4 - x^2$$

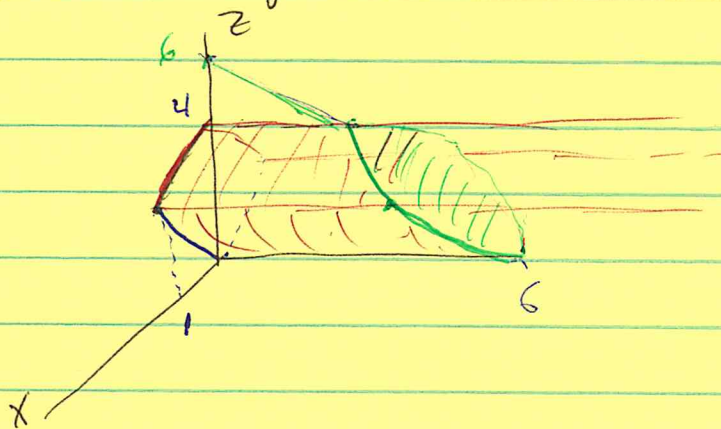
$$y = 0 \quad \leftarrow \text{xz-plane}$$

$$z + y = 6.$$

Sol Draw lots of pictures.



Slowly put them together



$$\text{Vol} = \int_{-1}^1 \int_{3x^2}^{4-x^2} \int_0^{6-z} 1 \, dy \, dz \, dx = \dots = \boxed{\frac{304}{15}}$$

Ex Find the volume bounded by

$$z = 3x^2 + y^2 + 7$$

$$y = x^2$$

$$z = 0$$

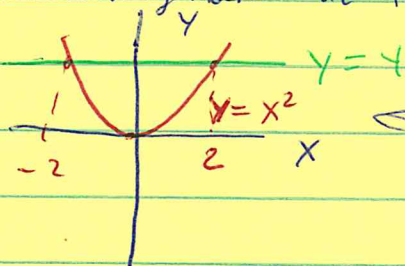
← xy -plane

$$y = 4$$

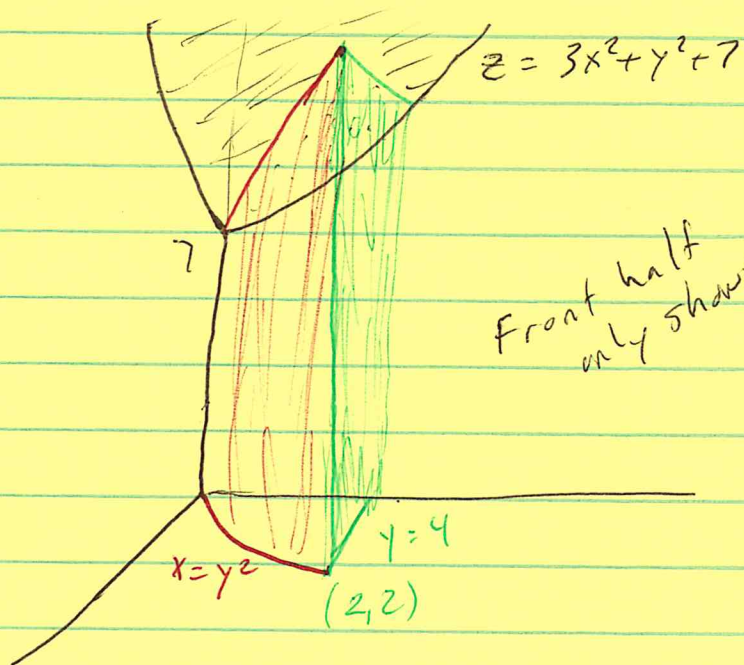
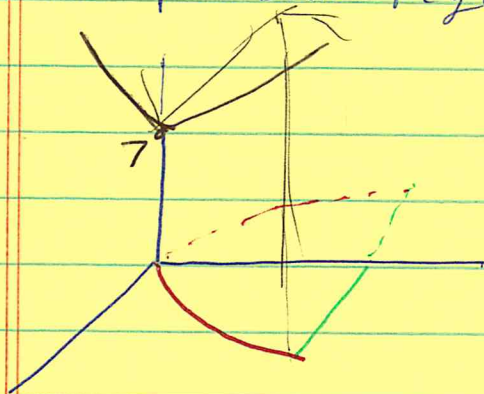
← a plane parallel to xz -plane

Sol We are going from the xy -plane ($z=0$) up to the paraboloid $z = 3x^2 + y^2 + 7$.

The equation $y = x^2$, $y = 4$ determine a bounded region in the xy -plane.



So, the solid region is below the paraboloid over the parabolic region in the xy -plane.



$$\text{Vol.} = \int_{-2}^2 \int_{x^2}^4 \int_0^{3x^2+y^2+7} 1 \, dz \, dy \, dx$$

$$= 2 \int_0^2 \int_{x^2}^4 (3x^2 + y^2 + 7) \, dy \, dx$$

using symmetry

$$= 2 \int_0^2 \left(3x^2 y + \frac{y^3}{3} + 7y \right) \Big|_{x^2}^4 \, dx$$

$$= 2 \int_0^2 \left(12x^2 + \frac{64}{3} + 28 \right) - \left(3x^4 + \frac{x^6}{3} + 7x^2 \right) \, dx$$

$$= 2 \int_0^2 \left(-\frac{x^6}{3} - 3x^4 + 5x^2 + \frac{148}{3} \right) \, dx$$

$$= 2 \left[-\frac{x^7}{7} - \frac{3x^5}{5} + \frac{5x^3}{3} + \frac{148}{3}x \right]_0^2$$

$$= 2 \left[-\frac{128}{7} - \frac{96}{5} + \frac{40}{3} + \frac{296}{3} \right] = \frac{18,208}{105}$$

$$\approx 173.409523$$

Ex Integrate $f(x, y, z) = xy + yz + xz$ over the solid ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} \leq 1.$$

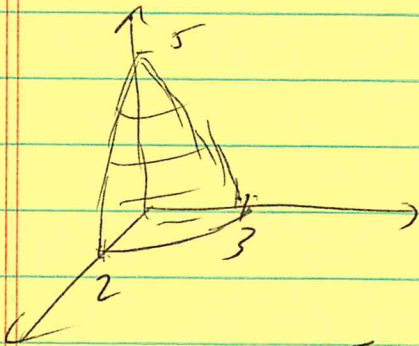
Sol Use symmetry. The answer is 0. Think about this. Compare to

$$\iint xy \, dx dy$$

over a disk or ellipse with center $(0,0)$.

Ex Find the volume of the solid ellipsoid above.

Sol



$$\text{Vol.} = 8 \int_0^5 \int_0^{3\sqrt{1-\frac{z^2}{25}}} \int_0^{2\sqrt{1-\frac{y^2}{9}-\frac{z^2}{25}}} 1 \, dx dy dz.$$

$$= 16 \int_0^5 \int_0^{3\sqrt{1-\frac{z^2}{25}}} \sqrt{1-\frac{y^2}{9}-\frac{z^2}{25}} \, dy dz$$

Use a trig sub. Treat z as a constant for now.

$$\text{Let } y = 3 \sqrt{1 - \frac{z^2}{25}} \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

$$\begin{aligned} \text{Then } \sqrt{1 - \frac{z^2}{25} - \frac{y^2}{9}} &= \sqrt{\left(1 - \frac{z^2}{25}\right) - \left(1 - \frac{z^2}{25}\right) \sin^2 \theta} \\ &= \sqrt{1 - \frac{z^2}{25}} \cos \theta. \end{aligned}$$

$$\text{Also, } dy = 3 \sqrt{1 - \frac{z^2}{25}} \cos \theta d\theta.$$

$$\text{Thus, } \text{Vol} = 16 \int_0^5 \int_0^{\pi/2} 3 \left(1 - \frac{z^2}{25}\right) \cos^2 \theta d\theta dz$$

$$\text{You check that } \int_0^{\pi/2} \cos^2 \theta d\theta = \frac{\pi}{4}.$$

$$= 12\pi \int_0^5 1 - \frac{z^2}{25} dz$$

$$\begin{aligned} &= 12\pi \left(z - \frac{z^3}{75} \right) \Big|_0^5 = 12\pi \left(5 - \frac{5^3}{75} \right) \\ &= 40\pi \end{aligned}$$

Note

The vol. inside the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

is

$$V = \frac{4}{3} abc\pi.$$

Applications of Triple Integrals

$$M = \text{Mass} = \iiint \rho(x, y, z) \, dV.$$

First Moments wrt to coordinate planes:

$$M_{yz} = \iiint x \rho \, dV$$

$$M_{xz} = \iiint y \rho \, dV$$

$$M_{xy} = \iiint z \rho \, dV$$

$$\text{Center of Mass: } \bar{x} = \frac{M_{yz}}{M}, \bar{y} = \frac{M_{xz}}{M}, \bar{z} = \frac{M_{xy}}{M}.$$

Second Moments wrt coordinate axes:

$$I_x = \iiint (y^2 + z^2) \rho \, dV$$

$$I_y = \iiint (x^2 + z^2) \rho \, dV$$

$$I_z = \iiint (x^2 + y^2) \rho \, dV$$

Radii of Gyration

$$\bar{\bar{x}} = \sqrt{\frac{I_x}{M}} \quad \bar{\bar{y}} = \sqrt{\frac{I_y}{M}} \quad \bar{\bar{z}} = \sqrt{\frac{I_z}{M}}$$

Ex Find the center of mass of the cube $[0, 1]^3$ with density $\rho(x, y, z) = xyz$.

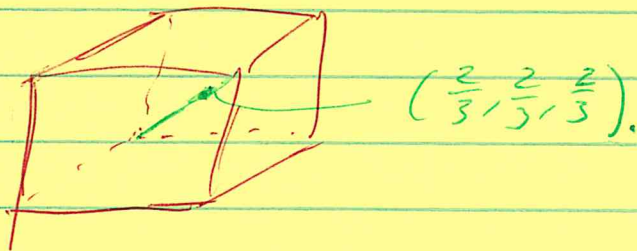
Sol $M = \int_0^1 \int_0^1 \int_0^1 xyz \, dx \, dy \, dz = \frac{1}{8}$

$$M_{yz} = \iiint x(xyz) \, dx \, dy \, dz = \frac{1}{12}$$

$$M_{xz} = \iiint y(xyz) \, dx \, dy \, dz = \text{same} = \frac{1}{12}$$

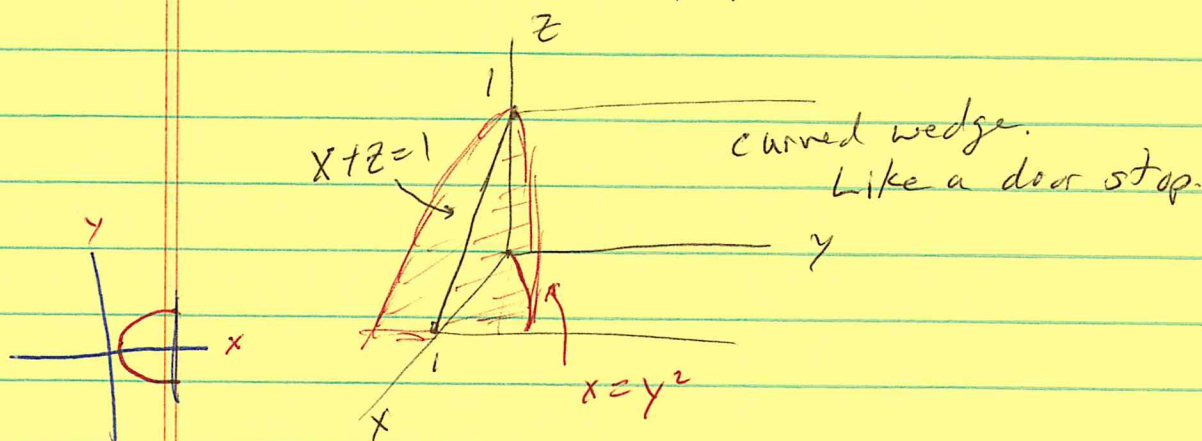
$$M_{xy} = \iiint z(xyz) \, dx \, dy \, dz = \text{same} = \frac{1}{12}$$

Thus $\bar{x} = \bar{y} = \bar{z} = \frac{\frac{1}{12}}{\frac{1}{8}} = \frac{8}{12} = \frac{2}{3}$



Ex Find the volume and centroid of the region R bounded by $x=y^2$, $z=0$, $x+z=1$.

Sol $z=0$ is the xy -plane.



$$V = \int_{-1}^1 \int_{y^2}^1 \int_0^{1-x} 1 \, dz \, dx \, dy = \dots = \frac{8}{15}$$

Centroid use $\rho=1$.

$$\begin{aligned} \bar{x} &= \frac{1}{V} \iiint x \, dV = \frac{15}{8} \int_{-1}^1 \int_{y^2}^1 \int_0^{1-x} x \, dz \, dx \, dy \\ &= \dots = \frac{3}{7} \end{aligned}$$

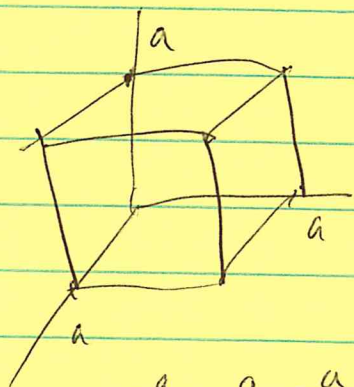
Clearly $\bar{y} = 0$.

$$\begin{aligned} \bar{z} &= \frac{1}{V} \iiint z \, dV = \frac{15}{8} \int_{-1}^1 \int_{y^2}^1 \int_0^{1-x} z \, dz \, dx \, dy \\ &= \dots = \frac{2}{7}. \end{aligned}$$

Centroid is $(\frac{3}{7}, 0, \frac{2}{7})$.

Ex (a) Find the moment of inertia and radius of gyration of a cube of edge length a with uniform density k with respect to an edge.

Sol



I'll use the z -axis as the edge.

$$\text{Mass} = \text{density} \times \text{vol.} = ka^3.$$

$$I_z = \int_0^a \int_0^a \int_0^a (x^2 + y^2) k \, dx \, dy \, dz$$
$$= k \int_0^a \int_0^a \left(\frac{a^3}{3} + y^2 a \right) dy \, dz$$

$$= k \int_0^a \frac{a^3}{3} a + \frac{a^3 a}{3} dz$$

$$= \frac{2ka^4}{3} \int_0^a dz = \frac{2ka^5}{3}$$

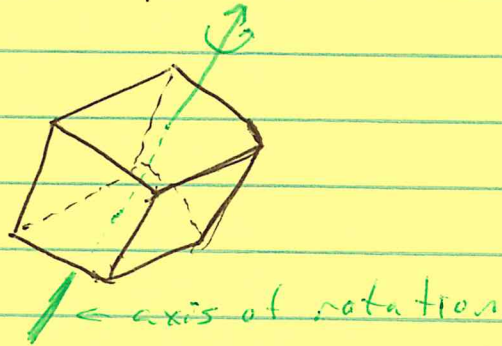
$$R_g \text{ or } \bar{z} = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{\frac{2ka^5}{3}}{ka^3}} = a\sqrt{\frac{2}{3}}$$

(b) If $a=1$ meter, $k=3 \text{ kg/m}^3$ and $\omega=3 \text{ rev/sec}$ what is the kinetic energy?

Sol

$$K.E. = \frac{1}{2} I_z \omega^2 = \frac{1}{2} \left(\frac{2 \cdot 3 \cdot 1^5}{3} \right) (3 \cdot 2\pi)^2 = \underline{36\pi^2 \approx 355.3 \text{ Joules}}$$

Ex (a) Find the moment of inertia and radius of gyration of a cube of edge length a with uniform density k with resp. to a line down the middle as shown



Sol

$$\text{Mass} = k a^3$$

$$I_z = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2 + y^2) k \, dx \, dy \, dz = \dots = \frac{k a^5}{6}$$

[z could go from 0 to a and give the same result.]

$$R_G = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{k a^5/6}{k a^3}} = \frac{a}{\sqrt{6}}$$

(b) If $a = 1$ meter, $k = 3 \text{ kg/m}^3$ and $\omega = 3 \text{ rev/sec}$ what is the kinetic energy?

$$\text{K.E.} = \frac{1}{2} I_z \omega^2 = \frac{1}{2} \left(\frac{3 \cdot 1^5}{6} \right) (3 \cdot 2\pi)^2 = \underline{9\pi^2 \approx 88.8 \text{ Joules}}$$