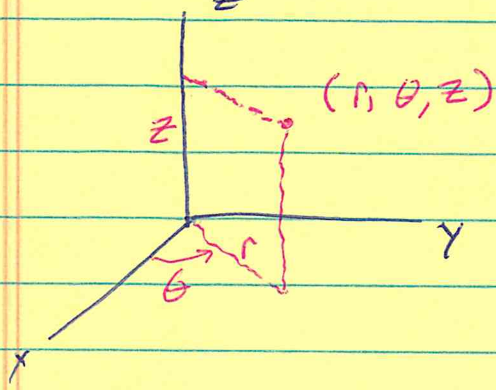
12.6 Cylindrical Coordinates



$r = \sqrt{x^2 + y^2}$

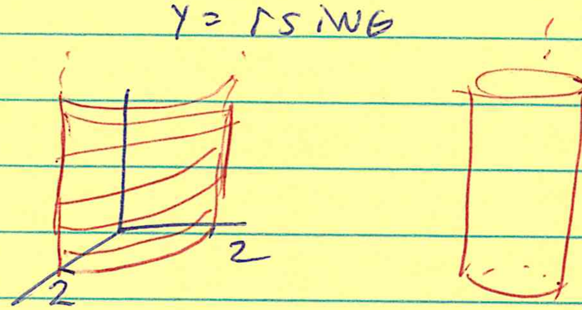
$\theta = \tan^{-1}(y/x)$ up to π

$z = z$

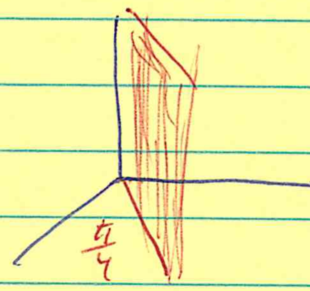
$x = r\cos\theta$

$y = r\sin\theta$

Ex Graph $r = 2$.



Ex Graph $\theta = \pi/4$.



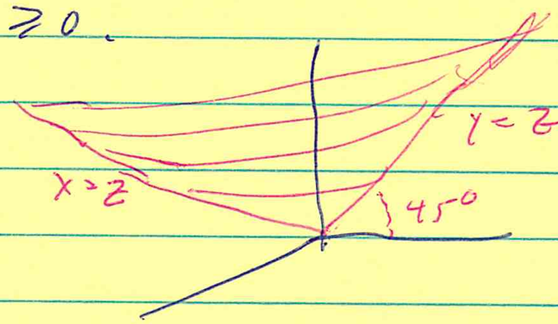
$\tan\theta = \tan\frac{\pi}{4} = 1$

$\frac{y}{x} = 1$

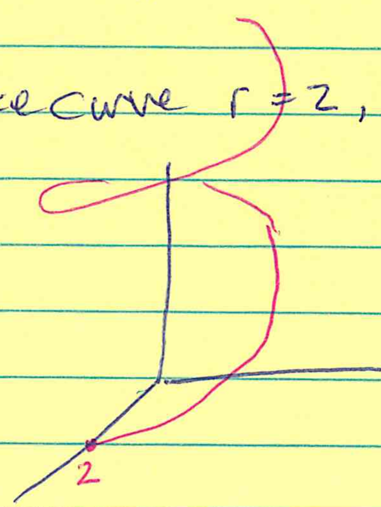
$y = x$

Ex Graph $r = z \geq 0$.

Cone



Ex Graph the space curve $r = 2$, $\theta = 2\pi t$, $z = t$.

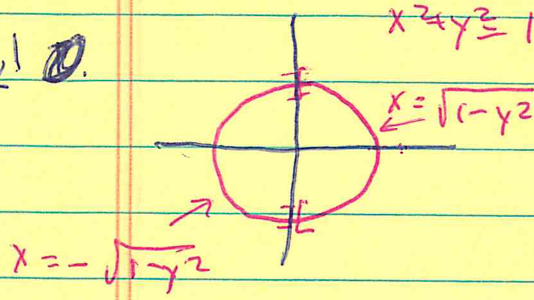


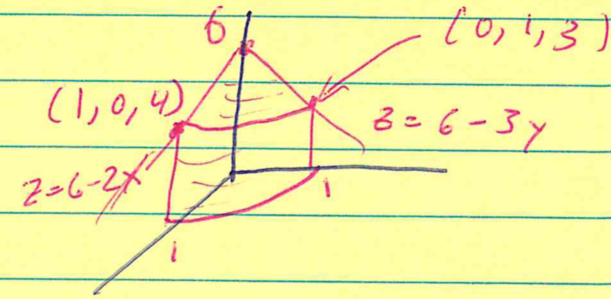
helix

Examples of triple integrals in cylindrical coords.

Ex 1 Find the volume under the plane $2x+3y+z=6$ and over the unit dist $x^2+y^2 \le 1$.

Sol 1.





In rect. coord's $V=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{6-2x-3y} 1 \, dz\, dx\, dy.$

But, in cyl. coord's

$$V=\int_0^{2\pi}\int_0^1\int_0^{6-2x-3y} 1 \, dz\, r\, dr\, d\theta.$$

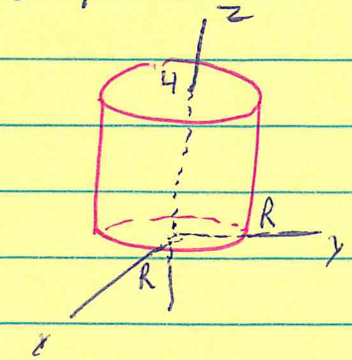
$$=\int_0^{2\pi}\int_0^1 (6-2x-3y)\, r\, dr\, d\theta$$

$$=\int_0^{2\pi}\int_0^1 6r-2r^2\cos\theta-3r^2\sin\theta \, dr\, d\theta$$

$$=\int_0^{2\pi} 3-\tfrac{2}{3}\cos\theta-\sin\theta\, d\theta$$

$$=\boxed{6\pi.}$$ $\longrightarrow$ = ave height × area of base.

Ex 2 Find the moment of inertia of a cylinder of radia R, height H and uniform density K with respect to it axis of symmetry and the radius of gyration.



Sol $\text{Mass} = K\pi R^2 H.$

$$I_z = \iiint (x^2+y^2)\, k\, dV$$

$$= k\int_0^H\int_0^{2\pi}\int_0^R r^2\, r\,dr\,d\theta\,dz$$

$$= k\frac{R^4}{4}2\pi H = \frac{1}{2}kR^4\pi H.$$

$$R_G = \bar{\bar{z}} = \sqrt{\frac{I_z}{\text{Mass}}} = \sqrt{\frac{kR^4\pi H}{2k\pi R^2 H}} = \frac{R}{\sqrt{2}}.$$

Ex 3 Repeat with density proportional to the square of the distance to the axis.

Sol. Let $\rho = kr^2$ be the density.

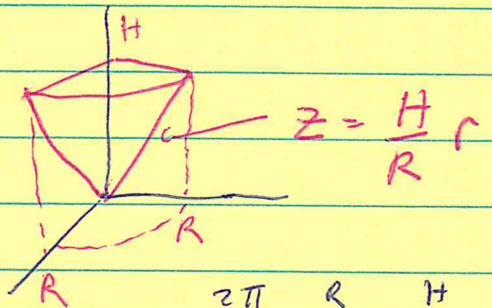
$$\text{Mass} = \int_0^H\int_0^{2\pi}\int_0^R kr^2\, r\,dr\,d\theta\,dz = \cdots\cdots = \frac{kR^4H\pi}{2}.$$

$$I_z = \int_0^H\int_0^{2\pi}\int_0^R r^2(kr^2)\, r\,dr\,d\theta\,dz = \cdots = \frac{kR^6H\pi}{3}$$

$$R_G = \sqrt{\frac{I_z}{\text{Mass}}} = \sqrt{\frac{kR^6H\pi/3}{kR^4H\pi/2}} = \sqrt{\frac{2}{3}}\,R.$$

Ex 4 Find the centroid of a cone of height and whose base has radius R.

Sol



By symmetry $\bar{x} = 0$, $\bar{y} = 0$.
We only need to find $\bar{z}$.

$$M_{xy} = \int_0^{2\pi} \int_0^R \int_{\frac{Hr}{R}}^{H} z \; r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^R \frac{z^2}{2} r \Big|_{z = \frac{Hr}{R}}^{z = H} dr$$

$$= \pi \int_0^R H^2 r - \frac{H^2}{R^2} r^3 \, dr$$

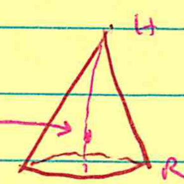
$$= \pi H^2 \left(\frac{R^2}{2} - \frac{1}{R^2} \frac{R^4}{4} \right) = \pi H^2 R^2 \left(\frac{1}{2} - \frac{1}{4} \right)$$

$$= \frac{\pi H^2 R^2}{4}.$$

$$V = \frac{H \pi R^2}{3}.$$

$$\bar{z} = \frac{M_{xy}}{V} = \frac{\pi H^2 R^2 / 4}{H \pi R^2 / 3} = \frac{3}{4} H.$$

But this is $\underline{\frac{1}{4} H}$ from the base!



Ex 5 Find the moment of inertial of a cone w.r.t. its axis where height $= H$, radius of base $= R$, density $= k$. If $H = R = k = 1$ and the cone is rotating 1 revolution per second what is its kenetic energy?

Sol. $$I_z = \iiint (x^2+y^2)\,k\,dV = k\int_0^{2\pi}\int_0^{R}\int_{\frac{Hr}{R}}^{H} r^2\, r\,dz\,dr\,d\theta$$

$$= 2\pi k\int_0^{R}\int_{\frac{Hr}{R}}^{H} r^3\,dz\,dr$$

$$= 2\pi k\int_0^{R} r^3 z\Big|_{\frac{Hr}{R}}^{H} dr = 2\pi k\int_0^{R} r^3\left(H - \frac{Hr}{R}\right) dr$$

$$= 2\pi k H\int_0^{R} r^3 - \frac{1}{R}r^4\,dr = 2\pi k H\left(\frac{R^4}{4} - \frac{1}{R}\frac{R^5}{5}\right)$$

$$= 2\pi k H R^4\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{\pi k H R^4}{10}$$

1 rps $= 2\pi$ rad/s

$$K.E. = \frac{1}{2} I_z \omega^2 = \frac{1}{2}\left(\frac{\pi 1\cdot 1\cdot 1^4}{10}\right)(2\pi)^2 = \frac{\pi^3}{5} \text{ Joules}$$

$$\approx 6.2\text{ J}$$