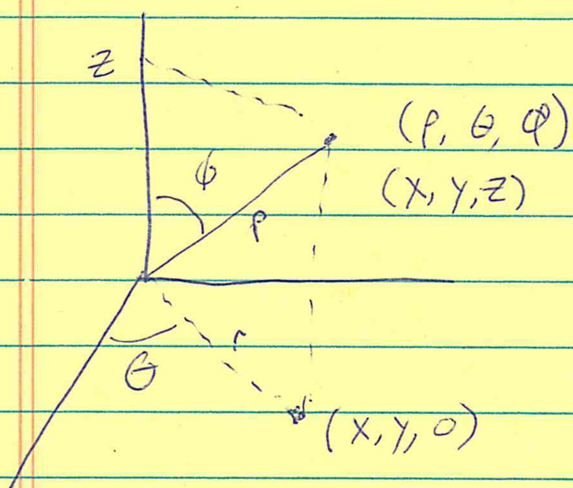
12.7 Spherical Coordinates



$z = \rho \cos \phi$

$r = \rho \sin \phi$

$x = r \cos \theta = \rho \sin \phi \cos \theta$

$y = r \sin \theta = \rho \sin \phi \sin \theta$

Ex $\rho = 2$ is a sphere of radius 2, center $(0, 0, 0)$.

Ex $\rho = 7 + \sin(3\phi) + \cos(2\theta)$ is a "bumby sphere."

Ex $\phi = \frac{\pi}{4}$ is a cone. $\phi = \frac{\pi}{2}$ is the xy-plane.

Ex Convert $x + y + z = 1$ into spherical coordinates

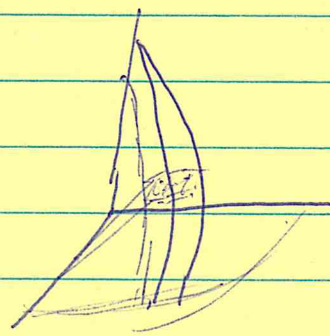
$$\rho \sin \phi \cos \theta + \rho \sin \phi \sin \theta + \rho \cos \phi = 1$$

$$\rho = \frac{1}{\sin \phi \cos \theta + \sin \phi \sin \theta + \cos \phi}$$

Spherical Block

$$\iiint \underline{\qquad\quad} \, dV$$

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



Cantelope

<u>Ex</u> Volume of a sphere.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$2\pi \frac{R^3}{3} \int_0^{\pi} \sin\phi \, d\phi = \frac{4}{3}\pi R^3$$

Ex Find the moment of inertia of a solid sphere with radius R and uniform density k, wrt a diameter. Find the radius of gyration.

Sol Mass = $\boxed{k \frac{4}{3}\pi R^3.}$

$$I_z = \iiint (x^2+y^2)k\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin^2\phi \; k\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \frac{2\pi k R^5}{5}\int_0^{\pi} \sin^3\phi\, d\phi.$$

$$= \frac{2\pi k R^5}{5}\int_0^{\pi} (1-\cos^2\phi)\sin\phi\, d\phi$$

$u = \cos\phi \quad du = -\sin\phi\, d\phi$

$$= -\frac{2\pi k R^5}{5}\int_1^{-1} 1-u^2\, du = \frac{2\pi k R^5}{5}\int_{-1}^{1} 1-u^2\, du$$

$$= \frac{2\pi k R^5}{5}\left[\left(u-\frac{u^3}{3}\right)\Big|_{-1}^{1}\right] = \frac{2\pi k R^5}{5}\left[\left(1-\frac{1}{3}\right)-\left(-1-\frac{-1}{3}\right)\right]$$

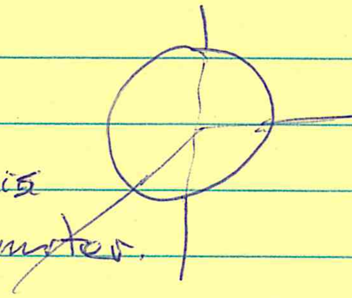
$$= \frac{2\pi k R^5}{5}\left(\frac{4}{3}\right) = \boxed{\frac{8\pi R^5 k}{15}}$$

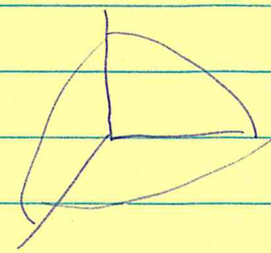
$$R_G = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{8\pi R^5 k/15}{\frac{4}{3}\pi R^3 k}} = \boxed{\sqrt{\frac{2}{5}}\, R}$$

Ex A solid sphere of radius R has density at each point proportional to its distance from a give diameter. Find the mass

Sol ~~$\rho(x,y,z) = kr$~~

Place the sphere so that the z-axis is the given diameter.





Then $\rho(x,y,z) = kr = k\rho \sin\phi.$

$$\text{Mass} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} k\rho\sin\phi \; \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi k \frac{R^4}{4} \int_0^{\pi} \sin^2\phi \, d\phi = 2\pi k \frac{R^4}{4} \cdot \frac{\pi}{2}$$

$$= \boxed{\frac{R^4\pi^2}{4} k}$$

#36 Convert to spherical coordinates and evalute

$$\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2z+y^2z+z^3)\, dz\,dx\,dy$$

Sol.

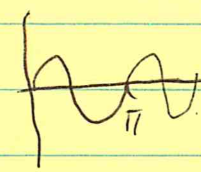
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} (x^2+y^2+z^2)z \quad \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int\int\int \rho^2 \rho\cos\phi \quad \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int\int\int \rho^5 \cos\phi \sin\phi \, d\rho\, d\phi\, d\theta$$

$$2\pi \frac{a^6}{6} \int_0^{\pi} \cos\phi \sin\phi \, d\phi$$

$$= 2\pi \frac{a^6}{6} \int_0^{\pi} \frac{1}{2}\sin(2\phi)\, d\phi = 0.$$



<u>Ex</u> Integrate $f(\rho, \theta, \phi) = e^{-\rho}$ over all of $\mathbb{R}^3$.

<u>Sol</u> $$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{-\rho}\,\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

I'll do θ and ϕ first.

$$\int_0^{\pi}\int_0^{2\pi} \rho^2 e^{-\rho} \sin\phi \, d\theta\, d\phi = 2\pi \int_0^{\pi} \rho^2 e^{-\rho} \sin\phi \, d\phi$$

$$= 4\pi \rho^2 e^{-\rho} \qquad \boxed{\int_0^{\pi} \sin\phi \, d\phi = 2}$$

$$\int_0^{\infty} \rho^2 e^{-\rho}\, d\rho = ? \qquad \text{Use integration by parts twice}$$

$$= -(2 + 2\rho + \rho^2) e^{-\rho} \Big|_0^{\infty} =$$

$$-\left(\lim_{\rho \to \infty} (2 + 2\rho + \rho^2) e^{-\rho} - (2) e^0 \right) =$$

Use L'Hop's Rule to show this limit is zero.

$= +2.$

Thus, the final answer is $4\pi \cdot 2 = \boxed{8\pi}$