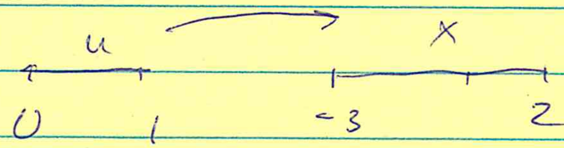
12.8 Change of Variables

Ex Consider $\int_{-3}^{2} f(x)\,dx$. We wish to transform it to an equivalent integral from 0 to 1. To do this we construct a one-to-one map from $[0,1]$ onto $[-3,2]$.



$$u \longmapsto 5u-3.$$

Let $x = 5u-3$. Then $dx = 5\,du$ and

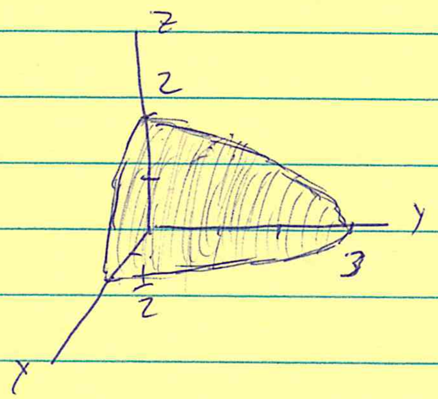
$$\int_{-3}^{2} f(x)\,dx = \int_{0}^{1} f(5u-3)\cdot 5\,du$$

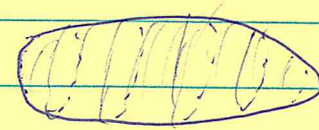
Note that $5 = \frac{dx}{du}$.

Ex Consider $\iiint_E f(x,y,z)\, dx\,dy\,dz = ☆$

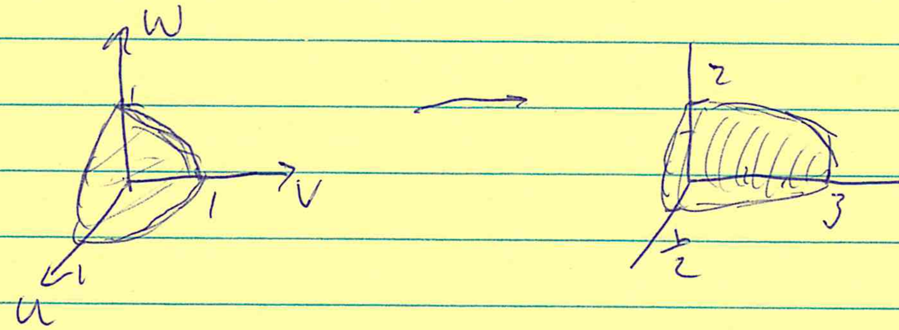
Where E is the solid ellipsoid with boundary

$$4x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1.$$





We will transform this into an equivalent integral over the unit sphere. To do this we need to find a map from $\mathbb{R}^3$ to $\mathbb{R}^3$ that take the unit sphere to our ellipsoid.



Consider the map $(u, v, w) \longrightarrow (\frac{u}{2}, 3v, 2w) = (x, y, z)$

That is

$$x = \frac{u}{2}$$
$$y = 3v$$
$$z = 2w.$$

Now, if $u^2+v^2+w^2=1$ then

$$(2x)^2+\left(\frac{y}{3}\right)^2+\left(\frac{z}{2}\right)^2=1$$

or $$4x^2+\frac{y^2}{9}+\frac{z^2}{4}=1.$$

So, this map, call it $T(u,v,w)$ takes the unit sphere to our ellipsoid. (It works on the points inside too)

Now $dx=\frac{1}{2}du$ Thus,
$dy=3\,dv$
$dz=2dw$.

$$\star=\iiint_E f(x,y,z)\,dx\,dy\,dz=\iiint_{\substack{\text{unit}\\\text{sphere}}} f\left(\frac{u}{2},3v,2w\right)3\,du\,dv\,dw$$

Now we could use spherical coordinates.

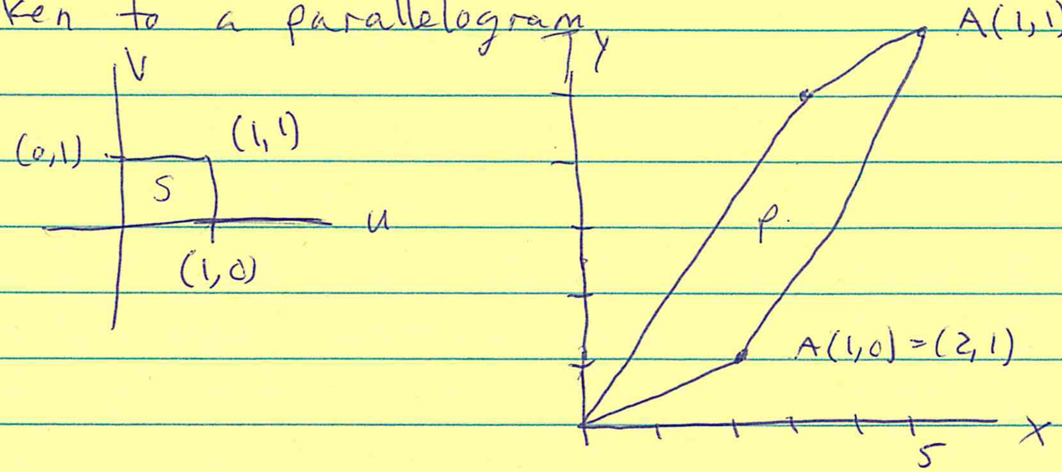
$$\star=3\int_0^{2\pi}\int_0^{\pi}\int_0^1 f\left(\frac{\rho\sin\phi\cos\theta}{2},3\rho\sin\phi\sin\theta,2\rho\cos\phi\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Ex Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and regard it as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$,

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2u+3v \\ u+5v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

That is $x = 2u + 3v$.
$y = u + 5v$.

Below shows ~~the~~ how the unit square $[0,1]\times[0,1]$ is taken to a parallelogram.



Now, suppose we want to integrate $f(x,y)$ over P. We can use a <u>change of variables</u> to transform the integral:

$$✩ = \iint_P f(x,y)\,dx\,dy = \int_0^1\int_0^1 f(2u+3v,\, u+5v)\ \underline{?}\ du\,dv.$$

What is ? The area of P is $|\det A| \cdot$ area of S. $\det A = 7$. Thus

$$✩ = \int_0^1\int_0^1 f(2u+3v,\, u+5v)\ 7\ du\,dv.$$

Det A is a measure of how areas in the (u,v) plane are strechted (or compressed).

Aslo, notice $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$

This matrix is called the <u>Jacobian</u> of the transformation.

For the ellipsoid example we could have defined the transformation as

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \frac{u}{2} \\ 3v \\ 2w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then the determinate is $\frac{1}{2} \cdot 3 \cdot 2 = 3$.

Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is not linear, but is one-to-one and is given by

$$(u, v) \to (g(u,v), h(u,v)) = (x, y).$$

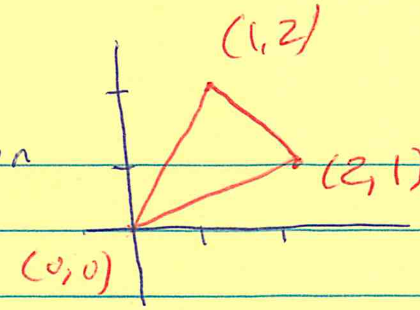
Then

$$J = \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{bmatrix}$$ is the Jacobian and

$$\iint_R f(x,y)\, dx\, dy = \iint_{T^{-1}(R)} f(g(u,v), h(u,v))\, |J|\, du\, dv.$$

The same thing is true if $T: \mathbb{R}^3 \to \mathbb{R}^3$.

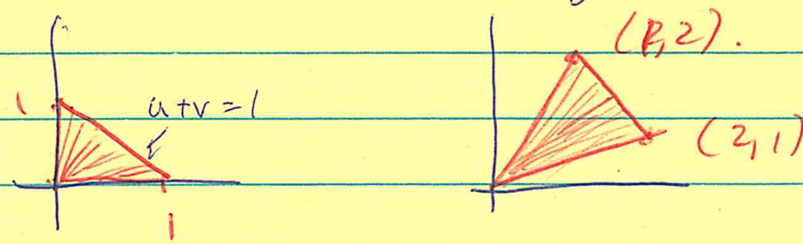
Ex Let R be the triangular region



Compute $\iint_R x-3y \, dA$

Hint: use $T(u,v) = (2u+v, u+2v)$.

Sol. The map T does the following



If we weren't given T as a hint, we could find it as follows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow a=2, c=1$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow b=1, d=2$$

So, we can think of T as $T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Then

$$\det T = 4-1 = 3.$$

Now

$$\iint_R x-3y\,dA = \int_0^1\int_0^{1-v} (2u+v)-3(u+2v)]3\,du\,dv$$

$$= -3\int_0^1\int_0^{1-v} u+5v\,du\,dv$$

$$= -3\int_0^1 \frac{u^2}{2}+5vu\Big|_0^{1-v}\,dv$$

$$= -3\int_0^1 \frac{(1-v)^2}{2}+5v(1-v)\,dv$$

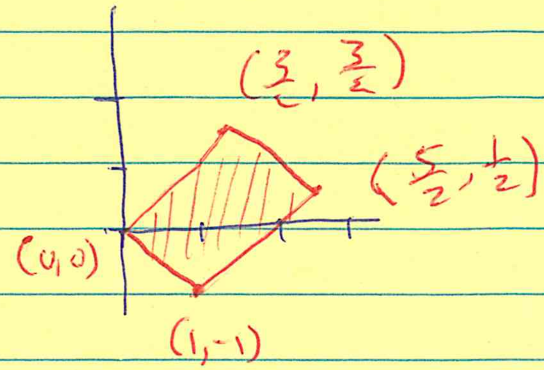
$$= -3\int_0^1 \frac{\overset{1-2v+v^2}{\cancel{1-3v+3v^2-v^3}}}{2} - 5v^2+5v\,dv$$

$$= -3\int_0^1 -\frac{9}{2}v^2+4v+\frac{1}{2}\,dv$$

$$= -3\left(-\frac{9}{2}\left(\frac{1}{3}\right)+4\left(\frac{1}{2}\right)+\frac{1}{2}\right)$$

$$= -3\left(-\frac{3}{2}+\frac{5}{2}\right) = -3(1) = \boxed{-3.}$$

Ex Compute $\iint_R (x+y)e^{x^2-y^2}\,dx\,dy$ where R is this rectangle



Sol The integral cannot be done in closed form. To rectify this we introduce to change of variables,

$$u = x+y \quad v = x-y.$$

Then $(x+y)e^{x^2-y^2} = (x+y)e^{(x+y)(x-y)} = ue^{uv}$, which is managable.

Now we need to find x and y in terms of u and v. This is not hard.

$$x = \frac{u+v}{2} \quad y = \frac{u-v}{2}.$$

Let $T = \begin{bmatrix} 1/2 & 1/2 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. Then

$$\iint_R (x+y)e^{x^2-y^2}\,dx\,dy = \iint_{T^{-1}(R)} ue^{uv}\,|\det T|\,\cancel{dx\,dy}\,dv\,du = \cancel{}$$

cusico

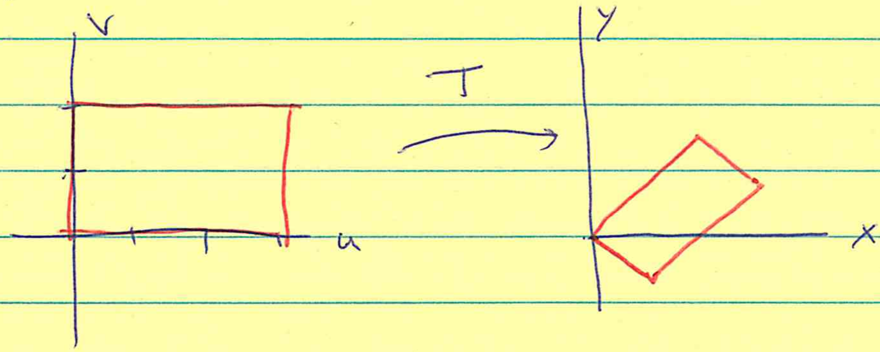
$\det T = -\frac{1}{2}$. But what is $T^{-1}(R)$?

T^{-1} takes $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} u \\ v \end{bmatrix}$, so $T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

We compute T^{-1} on the four corners of R.

$T^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad T^{-1}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\quad T^{-1}\begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$T^{-1}\begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Thus we have:



Now

$$\bigstar = \frac{1}{2}\int_0^3 \int_0^2 u e^{uv}\, dv\, du = \frac{1}{2}\int_0^3 e^{uv}\Big|_0^2 \, du$$

$$= \frac{1}{2}\int_0^3 e^{2v} - 1 \, dv = \frac{1}{2}\left(\frac{1}{2} e^{2v} - v \Big|_0^3 \right) =$$

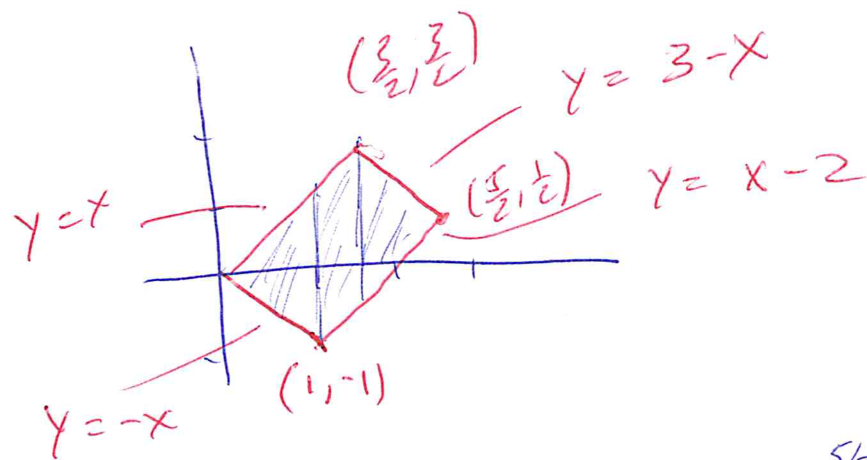
$$\frac{1}{2}\left[\left(\frac{1}{2} e^6 - 3 \right) - \left(\frac{1}{2} \right) \right] = \frac{e^6 - 7}{4}$$

Here I have done the original integral numerically. I had to divide the region R into three pieces.

```
> evalf( int(int((x+y)*exp(x^2-y^2),y=-x..x),x=0..1)+
         int(int((x+y)*exp(x^2-y^2),y=x-2..x),x=1..3/2)+
         int(int((x+y)*exp(x^2-y^2),y=x-2..3-x),x=3/2..5/2) );
                        99.10719837318378065209679513 58
```

Let's compare that to the result we got.

```
> evalf((-7+exp(6))/4);
                        99.10719837318378065209679513 60
>
```



$$\int_0^1\int_{-x}^{x} \_\_\_ \,dy\,dx + \int_1^{3/2}\int_{x-2}^{x} \_\_\_ \,dy\,dx + \int_{3/2}^{5/2}\int_{x-2}^{3-x} \_\_\_ \,dy\,dx$$