13.1 Vector Fields

$F: \mathbb{R}^2 \to \mathbb{R}^2$
or $F: \mathbb{R}^3 \to \mathbb{R}^3$

Ex 1 Let $F(x,y) = \langle -y, x \rangle$.

x-axis

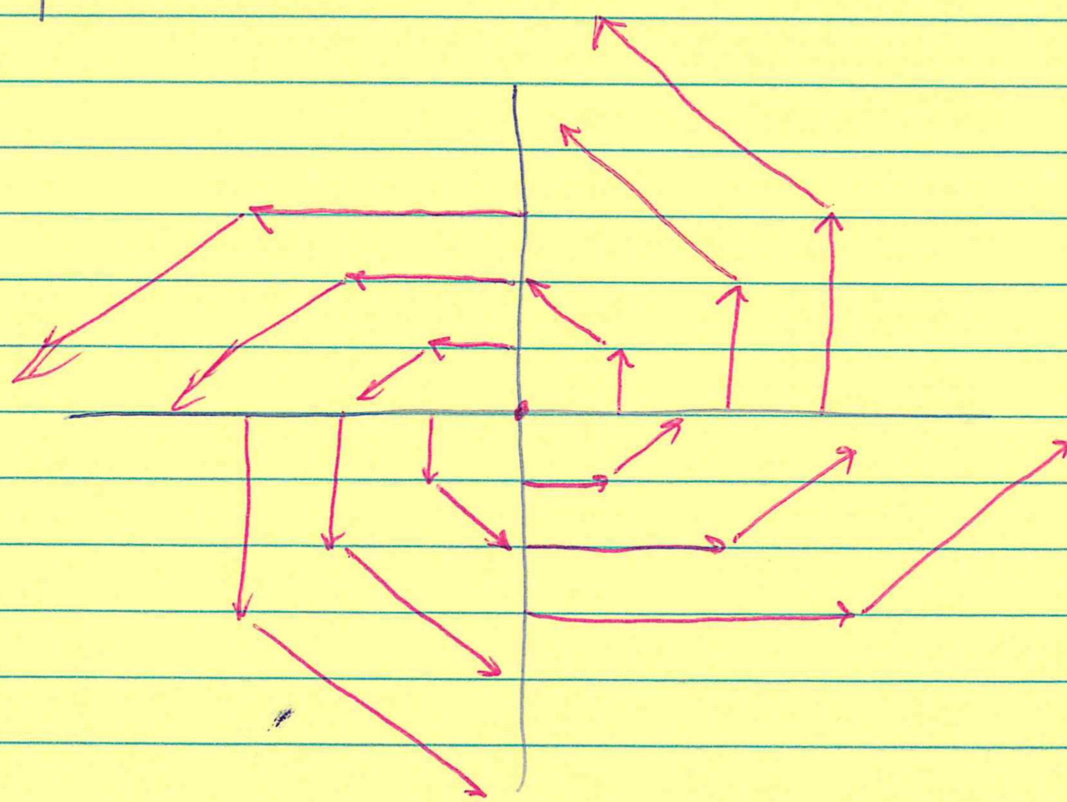
(x,y)	$\langle -y, x \rangle$
$(0,0)$	$\langle 0,0 \rangle$
$(1,0)$	$\langle 0,1 \rangle$
$(2,0)$	$\langle 0,2 \rangle$
$(3,0)$	$\langle 0,3 \rangle$
$(-1,0)$	$\langle 0,-1 \rangle$
$(-2,0)$	$\langle 0,-2 \rangle$

y-axis

$(0,1)$	$\langle -1,0 \rangle$
$(0,2)$	$\langle -2,0 \rangle$
$(0,3)$	$\langle -3,0 \rangle$
$(0,-1)$	$\langle 1,0 \rangle$
$(0,-2)$	$\langle 2,0 \rangle$
$\vdots$	$\vdots$

x=y

$(1,1)$	$\langle -1,1 \rangle$
$(2,2)$	$\langle -2,2 \rangle$
$(3,3)$	$\langle -3,3 \rangle$
$(-1,-1)$	$\langle 1,-1 \rangle$
$\vdots$	$\vdots$



Ex 2 Let $F(x,y) = \langle x-1, -\frac{y}{2} \rangle$

(x,y)	$\langle x-1, -\frac{y}{2} \rangle$
$(1,0)$	$\langle 0, 0 \rangle$
$(1,1)$	$\langle 0, -\frac{1}{2} \rangle$
$(1,2)$	$\langle 0, -1 \rangle$
$\vdots$	$\vdots$
$(1,-1)$	$\langle 0, \frac{1}{2} \rangle$
$(1,-2)$	$\langle 0, 1 \rangle$

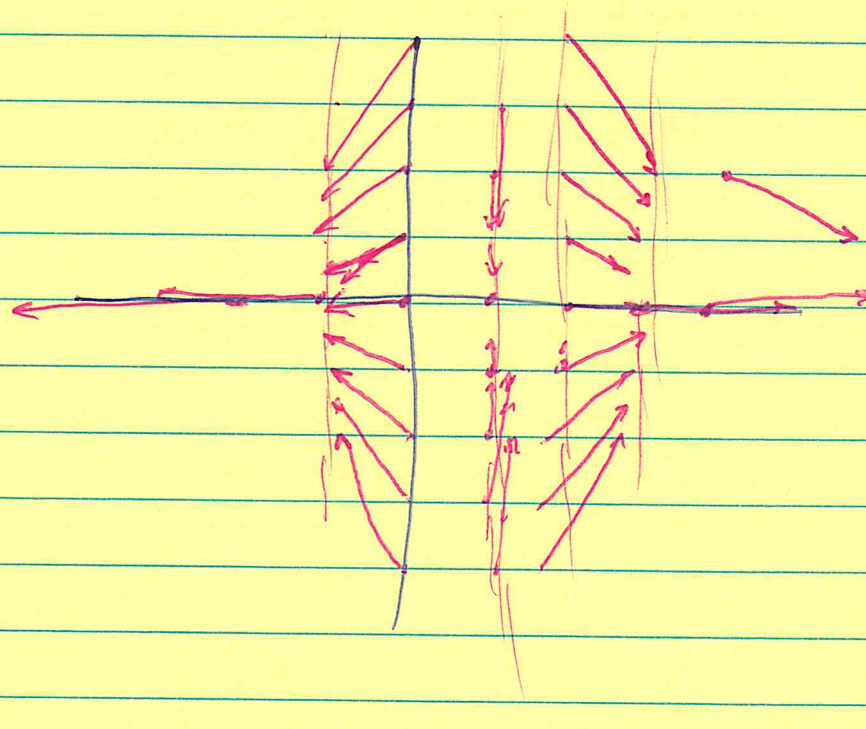
x-axis

(x,y)	$\langle x-1, -\frac{y}{2} \rangle$
$(0,0)$	$\langle -1, 0 \rangle$
$(1,0)$	$\langle 0, 0 \rangle$
$(2,0)$	$\langle 1, 0 \rangle$
$(3,0)$	$\langle 2, 0 \rangle$
$(-1,0)$	$\langle -2, 0 \rangle$
$(-2,0)$	$\langle -3, 0 \rangle$

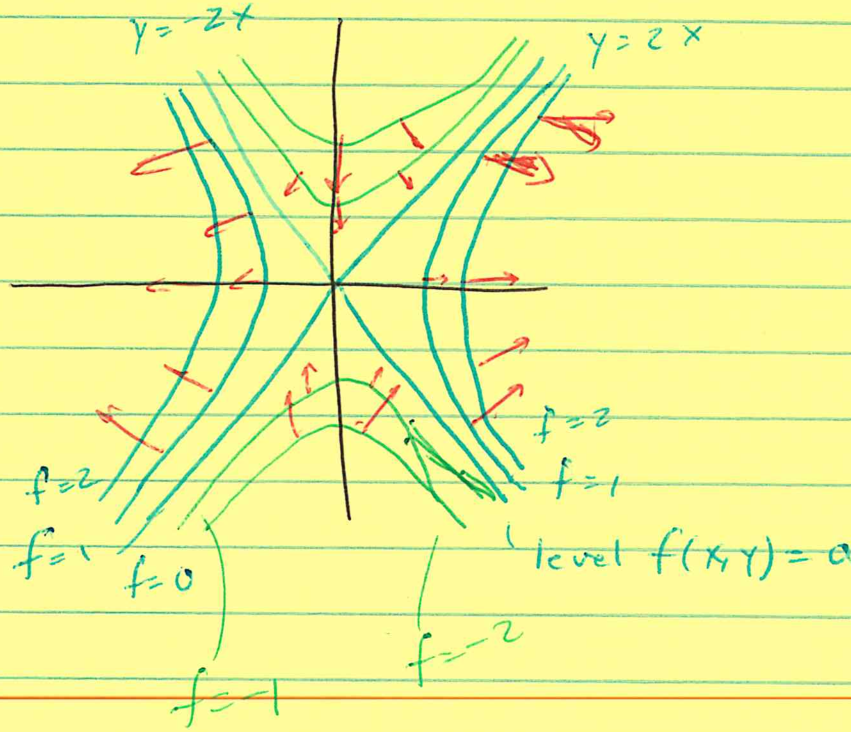
y-axis

(x,y)	$\langle x-1, -\frac{y}{2} \rangle$
$(0,1)$	$\langle -1, -\frac{1}{2} \rangle$
$(0,2)$	$\langle -1, -1 \rangle$
$(0,3)$	$\langle -1, -\frac{3}{2} \rangle$
$(0,4)$	$\langle -1, -2 \rangle$
$(0,-1)$	$\langle -1, \frac{1}{2} \rangle$
$(0,-2)$	$\langle -1, 1 \rangle$
$\vdots$	$\vdots$

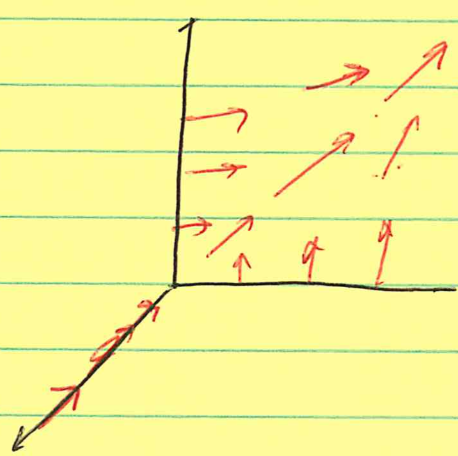
(x,y)	$\langle x-1, -\frac{y}{2} \rangle$
$(2,1)$	$\langle 1, -\frac{1}{2} \rangle$
$(2,2)$	$\langle 1, -1 \rangle$
$(2,3)$	$\langle 1, -\frac{3}{2} \rangle$
$(2,4)$	$\langle 1, -2 \rangle$
$(2,-1)$	$\langle 1, \frac{1}{2} \rangle$
$\vdots$	$\vdots$



Ex Let $f(x,y) = 4x^2 - y^2$. Let $F = \nabla f = \langle 8x, -2y \rangle$.
Then we can plot the level curves of $f(x,y)$ as a guide to the vector field given by its gradient.



Ex $F(x,y,z) = \langle -x, z, y \rangle$.



See the computer plots of these examples.

Def Let F be a vector field on $\mathbb{R}^2$ or $\mathbb{R}^3$. That is

$$F(x,y) = \langle P(x,y), Q(x,y) \rangle \quad \text{or}$$

$$F(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle.$$

Then there is a scalar function, $f(x,y)$ or $f(x,y,z)$, such that

$$F = \nabla f$$

we say F is a <u>conservative vector field</u>. (In your physics courses this refers to systems that have a potential energy function.)

Thm Let $F(x,y) = \langle P(x,y), Q(x,y) \rangle$. Then F is conservative iff

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

Partial Proof We prove one direction. Suppose that F is conservative. Suppose $F = \nabla f$. Then $P = f_x$ and $Q = f_y$. Thus, $Q_x = f_{yx}$ and $P_y = f_{xy}$. Since $f_{xy} = f_{yx}$ we have $Q_x = P_y$.

We will proof the other implication later in the course using Green's Thm. At the end of the course we will give a 3-dimensional version of this thm using Stokes' Thm.

We check the first two examples.

Ex 1 $F(x,y) = \langle -y, x\rangle$. $\frac{\partial -y}{\partial y} = -1$ and $\frac{\partial x}{\partial x} = 1$.

Since $-1 \neq 1$ this vector field is not conservative.

Ex 2 $F(x,y) = \langle x-1, -\frac{1}{2}y\rangle$, $\frac{\partial\, x-1}{\partial y} = 0$ and $\frac{\partial -\frac{1}{2}y}{\partial x} = 0$.

Since $0 = 0$, this v.f. is conservative.

In class I'll show how you can see this just from looking at the v.f. plots.

Ex Let $F(x,y) = \langle 2xy+1, x^2+3y^2\rangle$. Show that F is conservative and then find $f(x,y)$ such that $F = \nabla f$.

Sol Let $P = 2xy+1$ and $Q = x^2+3y^2$.

$\frac{\partial P}{\partial y} = 2x$, $\frac{\partial Q}{\partial x} = 2x$. Thus F is conservative.

We use "partial integration". Suppose $\nabla f = F$. Then $f_x = P$ and $f_y = Q$. Thus

$$f = \int P\,dx = \int 2xy+1\,dx = x^2y + x + C_1(y)$$

and

$$f = \int Q\,dy = \int x^2+3y^2\,dy = x^2y + y^3 + C_2(x).$$

Hence $f(x,y) = x^2y + x + y^3$ will work. Check this!

Here are some examples for you to check. For each determine if the vector field is conservative. If it is, find a potential function, that is find $f(x,y)$ such that $F = \nabla f$.

1. $F = \langle 3xy,\ y^2 + e^x \rangle$

2. $F = \langle 2xy^2 + 1,\ 2x^2y + 3y^2 \rangle$

3. $F = \langle 3x^2y + ye^{xy},\ x^3 + xe^{xy} \rangle$

4. $F = \langle x^2 + y^2,\ \sin(xy) \rangle$

5. $F = \langle e^x \cos y,\ -e^x \sin y \rangle$.

Answers are on the next page.

1. Not conservative.

2. Conservative. $f(x,y) = x + y^3 + x^2y^2$

3. Conservative. $f(x,y) = x^3y + e^{xy}$.

4. Not conservative.

5. Conservative. $f(x,y) = e^x \cos y$.