13.2

Line Integrals

These really should be called path or curve integrals.

Recall arc length: let C be a curve and let $r(t)$ be a parameterization. Then

speed × time = distance.

$$\text{length of } C = \int_C ds = \int_a^b |r'(t)|\, dt.$$

$$= \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

Let $f(x,y,z)$ be a function (Maybe it gives the density (mass/unit length) of a cord or wire lying along the path C.

Then the line integral of f along C is

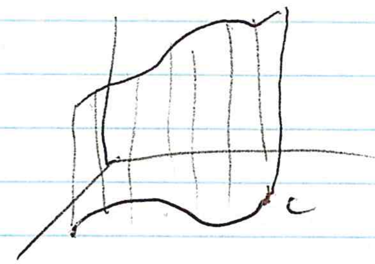
$$\int_C f\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

where $r(t) = \langle x(t), y(t), z(t) \rangle$ is a parameterization of C.

If $f(x,y)$ is a function of two varibles and C is in the xy-plane this becomes

$$\int_C f\, ds = \int_a^b f(x(t), y(t)) \sqrt{(x')^2 + (y')^2}\, dt.$$

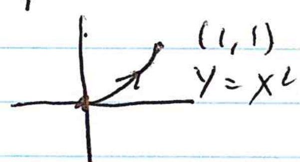
Here it could be that $z=f(x,y)$ is the height above the curve C and the integral of f along C is an area.



<u>Note</u> The result is independent of the parameterization of C unless you reverse the direction. In that case the sign changes.

$$\int_{C_1} f\,ds = -\int_{C_2} f\,ds.$$

<u>Ex</u> Let C be the parabolic curve in $\mathbb{R}^2$ shown:



Assume it starts at (0,0) and ends at (1,1).
Let $f(x,y) = 8y/x$. Find $\int_C f\,ds$.

Solution Step 1: Find a parameterization.

Let $x = t$ and $y = t^2$ $t \in [0,1]$.

Step 2 Set up integral.

$$\int_c f\,ds = \int_0^1 \frac{8t^2}{t}\sqrt{1^2+(2t)^2}\,dt.$$

$$= \int_0^1 8t\sqrt{1+4t^2}\,dt.$$

Step 3 Do the integration.

Let $u = 1+4t^2$. Then $du = 8t\,dt$
and u ranges from 1 to 5.

$$\int_1^5 (u)^{\frac{1}{2}}\,du = \frac{2}{3}\left(5^{3/2}-1\right) = \frac{2}{3}\left(5\sqrt{5}-1\right)$$

$$\approx 6.786893258$$

Ex Let C be the line segment starting at $(1,2,3)$ and ending at $(2,4,7)$. Let $f(x,y,z) = x^2 - yz$. Compute $\int_C f\, ds$.

Solution Step 1: Parametrize C,

Let $r(t) = \langle 1, 2, 4\rangle t + \langle 1, 2, 3\rangle$ for $0 \le t \le 1$.

Note $r'(t) = \langle 1, 2, 4\rangle$.

$x(t) = t+1$
$y(t) = 2t+2$
$z(t) = 4t+3$

Step 2 step up integral.

$$\int_C f\, ds = \int_0^1 \Big[\underbrace{(t+1)^2}_{x^2} - \underbrace{(2t+2)}_{y}\underbrace{(4t+3)}_{z}\Big]\sqrt{1+4+16}\; dt$$

$$= \sqrt{21}\int_0^1 (t^2+2t+1) - (8t^2+14t+6)\; dt$$

$$= \sqrt{21}\int_0^1 (-7t^2 - 12t - 5)\; dt$$

$$= -\sqrt{21}\left(\frac{7}{3} + \frac{12}{2} + 5\cdot 1\right) = -\sqrt{21}\left(\frac{14+36+30}{6}\right)$$

$$-\frac{80}{6}\sqrt{21} = -\frac{40}{3}\sqrt{21} \approx -61.10100927.$$

Ex Let C be the unit circle in $\mathbb{R}^2$ centered at $(0,0)$, oriented clockwise, starting and ending at $(1,0)$.

Let $f = 2x - 3y$. Compute $\oint_C f\, ds$.

Solution Step 1: parameterize C.

Let $x(t) = \cos t$ and $y(t) = -\sin t$.

$r(t) = \langle \cos t, -\sin t \rangle, \quad 0 \le t \le 2\pi.$

Step 2 Set up integral.

$r'(t) = \langle -\sin t, -\cos t \rangle$

$|r'(t)| = \sqrt{(\quad)^2 + (\quad)^2} = 1.$

$$\int_C f\, ds = \int_0^{2\pi} (2\cos t + 3\sin t)(1)\, dt = 0.$$

Line Integrals through vector fields

Def Let $F(x,y,z)$ be a vector field.

Let C be a curve in $\mathbb{R}^3$. Let $r(t)$ be a parameterization of C. Then the line integral of F along C is

$$\int_C \mathbf{F}\cdot T\, ds = \int_C F\cdot dr = \int_a^b F\cdot \frac{dr}{dt}\, dt$$

$$= \int F(x(t), y(t), z(t)) \cdot \langle x'(t), y'(t), z'(t)\rangle\, dt.$$

Notation/Motivation

$$\underbrace{\underbrace{F\cdot T}_{\text{comp of } F \text{ tangent to curve.}}\, ds}_{=\text{ work done along } ds} = F\cdot\frac{r'}{|r'|}\,|r'|\,dt = F\cdot r'\,dt.$$

Book likes to write $F\cdot\frac{dr}{dt}\,dt$ as $F\cdot dr$.

Compare to 7.~~5~~6, first subsection on <u>work</u>.

Ex Let C be given by $r(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$.

Let $F(x,y,z) = \langle xy, y^2, z \rangle$.

Compute $\int_C F \cdot dr$, i.e. find the work in moving along C.

Solution $r'(t) = \langle 1, 2t, 3t^2 \rangle$

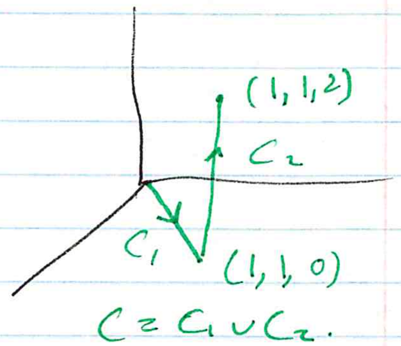
$F(t) = \langle t^3, t^4, t^3 \rangle$

$F \cdot r' = t^3 + 2t^5 + 3t^5 = t^3 + 5t^5$.

$\int_C F \cdot dr = \int_0^1 t^3 + 5t^5 \, dt = \frac{1}{4} + \frac{5}{6} = \frac{13}{12}$.

Ex Let $F = \langle 3y, 2x+z, xy \rangle$ be a force field.

How much work is done in moving a partical along the path shown:



Solution We will do this in 2 parts.

Let $r_1(t) = \langle t, t, 0 \rangle \quad 0 \le t \le 1$.

Let $r_2(t) = \langle 1, 1, 2t \rangle \quad 0 \le t \le 1$.

$$\int_C F \cdot dr = \int_{C_1} F \cdot dr_1 + \int_{C_2} F \cdot dr_2 .$$

$$\int_{C_1} F \cdot dr_1 = \int_0^1 \langle 3t, 2t, t^2 \rangle \cdot \langle 1, 1, 0 \rangle \, dt$$

$$= \int_0^1 5t \, dt = 5/2 .$$

$$\int_{C_2} F \cdot dr_2 = \int_0^1 \langle 3, 2+2t, 1 \rangle \cdot \langle 0, 0, 2 \rangle \, dt$$

$$= \int_0^1 2 \, dt = 2.$$

Thus, $\int_C F \cdot dr = \frac{5}{2} + 2 = 4.5$

Differential Notation

$$\int_C F\cdot dr = \int_a^b F\cdot \frac{dr}{dt}\,dt =$$

$$\int_c^b \langle M, N, P\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle dt$$

$$= \int_a^b \left(M\,\frac{dx}{\cancel{dt}} + N\cdot\frac{dy}{\cancel{dt}} + P\,\frac{dz}{\cancel{dt}}\right)\cancel{dt}$$

$$= \int_C M\,dx + N\,dy + P\,dz$$

This notation is common in engineering textbooks. I personally do not like it.

Ex This example is in the plane. Let C be the line segment in $\mathbb{R}^2$ starting at $(1,2)$ and ending at $(3,7)$.

Compute: $\displaystyle\int_C x^2 y\,dx + yx^3\,dy.$

Solution Parametrize C. $r(t) = \langle 2,5\rangle t + \langle 1,2\rangle,\ 0\le t\le 1.$

So $x(t) = 2t+1 \quad y(t) = 5t+2.$

$$\int_C x^2 y\,dx + yx^3\,dy = \cancel{\int F\cdot \frac{dr}{dt}\,dt}$$

$$= \int_0^1 (2t+1)^2(5t+2)\underbrace{2}_{\frac{dx}{dt}}\,dt + \int_0^1 (5t+2)(2t+1)^3\underbrace{5}_{\frac{dy}{dt}}\,dt$$

=

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> int((2*t+1)^2*(5*t+2)*2+(5*t+2)*(2*t+1)^3*5,t=0..1);
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$$\frac{1939}{6}$$