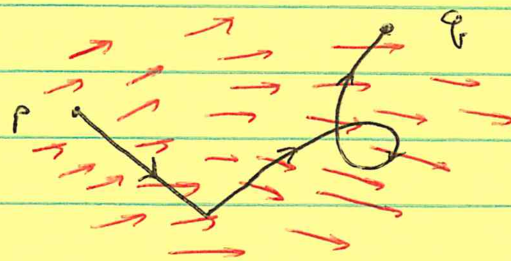
13.3 Path Independence and the Fundamental Thm of Line Integrals

Thm (FTLI) Let C be a piecewise smooth path with starting point $p = (x_1, y_1, z_1)$ and ending point $q = (x_2, y_2, z_2)$. Let F be a vector field with continuous gradient in an open region containing C.

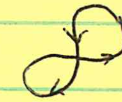


If F is conservative then

$$\int_C F \cdot T \, ds = \int_C \nabla f \cdot T \, ds = f(q) - f(p)$$

where $f(x, y, z)$ is a potential function for F, $F = \nabla f$.

This means that it does not matter what path we use to get from p to q. In particular, for any closed loop C



is a loop.

$$\oint F \cdot T \, ds = 0$$

Pf: Assume for now that C is smooth and that $r(t)$ is a parameterization for C.

$$\int_C F \cdot T \, ds = \int_a^b \nabla f \cdot \frac{dr}{dt} \, dt = \int_a^b f_x x' + f_y y' + f_z z' \, dt$$

But
$$\frac{d \, f(x(t), y(t), z(t))}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$
$$= f_x x' + f_y y' + f_z z'.$$

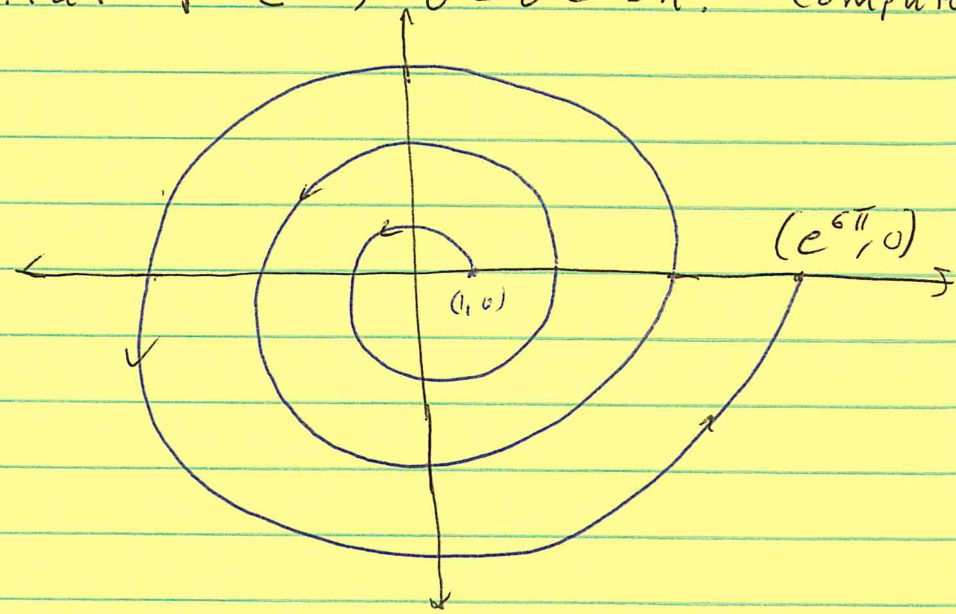
Thus,

$$\int_C F \cdot T \, ds = \int_a^b \frac{d}{dt} f(x(t), y(t), z(t)) \, dt$$
$$= f(x(t), y(t), z(t)) \Big|_a^b = f(x(b), y(b), z(b)) - f(x(a), y(b), z(b))$$
$$= f(q) - f(p).$$

$\blacksquare$

Can generalize to piecewise smooth curves one piece at a time.

Ex Let $F(x,y) = \langle 3y, 3x+y \rangle$. Let C be the spiral $r = e^{\theta}$, $0 \le \theta \le 6\pi$. Compute $\int_C F \cdot T \, ds$.



We will do this problem 4 ways.

Method I (Old Way) We parameterize the curve. I'll use θ as the parameter.

$$r(\theta) = \langle r\cos\theta, r\sin\theta \rangle = \langle e^{\theta}\cos\theta, e^{\theta}\sin\theta \rangle.$$

Now $$\int_C F \cdot T \, ds = \int_0^{6\pi} F \cdot \frac{dr}{d\theta} \, d\theta.$$

$$r'(\theta) = \langle e^{\theta}\cos\theta - e^{\theta}\sin\theta, \; e^{\theta}\sin\theta + e^{\theta}\cos\theta \rangle$$

$$F(\theta) = \langle 3e^{\theta}\sin\theta, \; 3e^{\theta}\cos\theta + e^{\theta}\sin\theta \rangle$$

$$F \cdot r' = e^{2\theta}\left(3sc - 3s^2 + 3sc + 3c^2 + s^2 + sc\right)$$

Thus,

$$\int_C F \cdot T \, ds =$$

$$\int_0^{6\pi} e^{2\theta} \left(7sc - 2s^2 + 3c^2 \right) d\theta$$

Computer integration gives 0.

Thus, $\int_C F \cdot T \, ds = 0$.

Method II Check to see if the vector field is conservative.

$\frac{\partial}{\partial y}(3y) = 3 \quad \frac{\partial}{\partial x}(3x+y) = 3$. Hence conservative.

Now find $f(x,y)$ with $F = \nabla f$.

$f = \int 3y \, dx = 3xy + C_1(y)$

$f = \int 3x+y \, dy = 3xy + \frac{1}{2}y^2 + C_2(x)$.

Let $f(x,y) = 3xy + \frac{1}{2}y^2$. You can check this works.

By the FTLI $\int_C F \cdot T \, ds = f(e^{6\pi}, 0) - f(1,0) = 0 - 0 = 0$.

Method III   Since F is conservative it is path independent. So, we can use an easier path. and not have to find f.

Let $r(t) = \langle (e^{6\pi}-1)t+1,\ 0 \rangle.$ $0 \le t \le 1$

Now $r'(t) = \langle e^{6\pi}-1,\ 0 \rangle$

and $F(t) = \langle 3\cdot 0,\ 3(e^{6\pi}-1)t+3+0 \rangle$
$= \langle 0,\ 3(e^{6\pi}-1)t+3 \rangle.$

$$F\cdot r' = 0\cdot(e^{6\pi}-1)+3\cdot 0 = 0.$$

Thus $\int F\cdot T\,ds = \int_0^1 F\cdot r'\,dt = \int_0^1 0\,dt = 0.$

Method IV   Look at the picture. The path from III is $\perp$ to the vector field. Thus $\int F\cdot T\,ds = 0.$

Ex Let $F = \langle 2x+z, 2y, x \rangle$.
Let C be the path $r(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$.
Find $\int_C F \cdot T \, ds$.

Sol I

$$\int_C F \cdot T \, ds = \int_0^1 F \cdot \frac{dr}{dt} \, dt = \int_0^1 \langle 2t+t^3, 2t^2, t \rangle \cdot \langle 1, 2t, 3t^2 \rangle \, dt$$

$$= \int_0^1 2t + t^3 + 4t^3 + 3t^3 \, dt$$

$$= \int_0^1 2t + 8t^3 \, dt = 2 \cdot \tfrac{1}{2} + 8 \cdot \tfrac{1}{4} = 1 + 2 = 3.$$

Sol II I claim F is conservative. We do not have a test for this (yet) but if I can find a potential function, then F is conservative.

$$f = \int 2x+z \, dx = x^2 + xz + C_1(y,z)$$
$$f = \int 2y \, dy = y^2 + C_2(x,z)$$
$$f = \int x \, dz = xz + C_3(x,y)$$

Let $f(x,y,z) = x^2 + y^2 + xz$. Check that $\nabla f = F$.

By the FTLI $\int_C F \cdot T \, ds = f(1,1,1) - f(0,0,0) = 3 - 0 = 3.$

Ex Let C be the unit circle, $x^2+y^2=1$, counter clockwise starting at $(0,1)$. Compute

$$\oint_C (2xy)\,dx + (x^2+2x)\,dy.$$

Sol **Step 1** Check if conservative. If the vector field $\langle 2xy,\ x^2+2x\rangle$ is cons. the answer is 0.

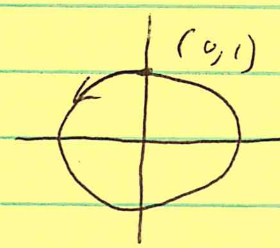
$$\partial_y(2xy) = 2x, \quad \partial_x(x^2+2x) = 2x+2.$$

It is not cons. but it almost is! Watch.

$$\oint_C (2xy)\,dx + (x^2+2x)\,dy = \underbrace{\oint_C (2xy)\,dx + (x^2)\,dy}_{=0} + \oint_C 2x\,dy$$

So, we only need to compute $\oint_C 2x\,dy$.

Step 2 Parametrize C.

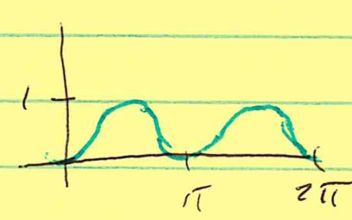


Use $r(t) = \langle -\sin t, \cos t\rangle$

Step 3 Do the integration.

$$\oint_C 2x\,dy = \int_0^{2\pi} 2\,x(t)\,\frac{dy}{dt}\,dt = \int_0^{2\pi} (-2\sin t)(-\sin t)\,dt$$

$$= 2\int_0^{2\pi} \sin^2 t\,dt = 2\pi$$



Thm The 3 following conditions are equivalent.

1. F is conservative

2. $\int_C F \cdot T\, ds$ or $\int_C F \cdot dr$ or $\int_a^b F \cdot \frac{dr}{dt}\, dt$

only dependent on the end points. Specifically

$$\int_C F \cdot T\, ds = f(b) - f(a) \quad \text{where } F = \nabla f.$$

3. $\oint_C F \cdot T\, ds = 0$ for all closed curves.