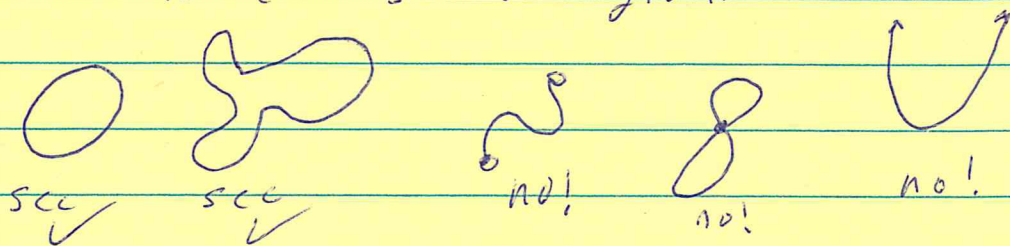


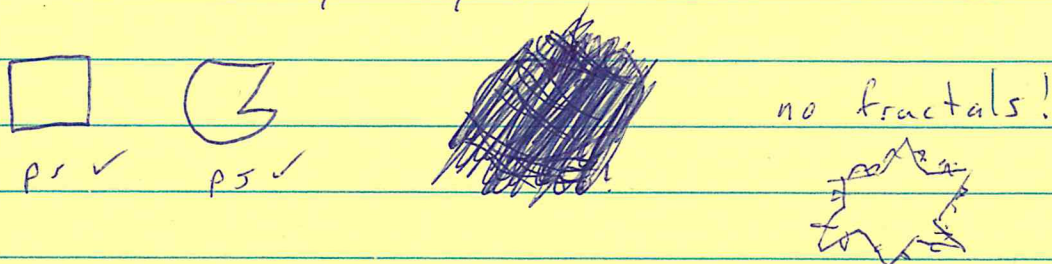
13.4

Green's Thm

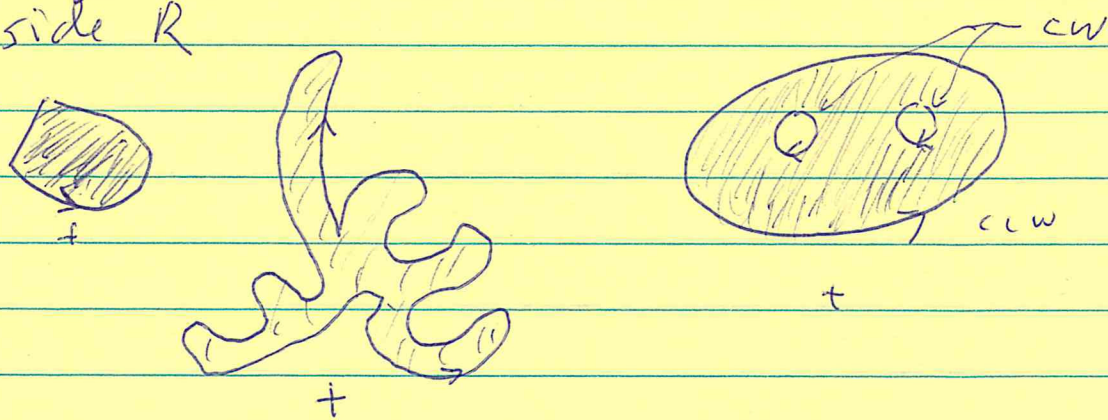
Def A curve in \mathbb{R}^2 is a simple closed curve, scc, if it is finite, has no end points, no self intersection points and enclosed a region.



A curve is piecewise smooth^{PS} if it is a union of finitely many differentiable curves.



Let R be a finite (bounded) region in \mathbb{R}^2 whose boundary is a finite collection of ps scc's: $\partial R = C_1 \cup C_2 \cup \dots \cup C_n$. A curve in ∂R is said to ~~be~~ have a positive orientation if when facing with the arrow your left hand would be inside R .



Green's Thm Let R be a bounded region of \mathbb{R}^2 with $\partial R = C_1 \cup C_2 \cup \dots \cup C_n$, a finite collection of ps $s \subset \mathbb{C}$, oriented positively. Let $F = \langle M, N \rangle$ be a vector field with continuous partial derivatives in R . Then

$$\oint_{\partial R} F \cdot T ds = \iint_R N_x - M_y dA$$

This is also written

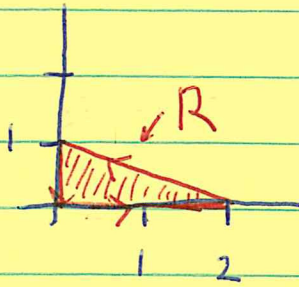
$$\oint_{\partial R} M dx + N dy = \iint_R N_x - M_y dA$$

Rmk's If $N_x = M_y$ then $\oint_{\partial R} F \cdot T ds = 0$, proving the claim we made about the test for conservative vector fields.

Compare Green's Thm to the FTC and the FTL Int.

There is a 3D version of Green's Thm called Stoke's Thm that we will cover later.

Ex 1



$$\text{Let } F = \langle -2y + x, x + y \rangle.$$

$$\text{Find } \oint_{\partial R} F \cdot T \, ds.$$

Sol

We could break up ∂R into 3 line segments and do three line integrals. Instead we will use Green's Thm.

$$\partial_y(-2y + x) = -2 \quad \partial_x(x + y) = 1.$$

$$\begin{aligned} \text{Thus, } \oint_{\partial R} F \cdot T \, ds &= \iint_R 1 - (-2) \, dA = 3 \iint_R dA \\ &= 3 (\text{area of } R) = 3 \left(\frac{1}{2} \cdot 2 \cdot 1 \right) = \mathbf{3}. \end{aligned}$$

Ex 2

Compute $\oint_c e^x \sin(2y) \, dx + 2e^x \cos(2y) \, dy$, where c is the circle $x^2 + y^2 = 1$, ccw.

Sol

The vector field is

$$F = \langle e^x \sin(2y), 2e^x \cos(2y) \rangle.$$

$$\partial_y(e^x \sin(2y)) = 2e^x \cos(2y)$$

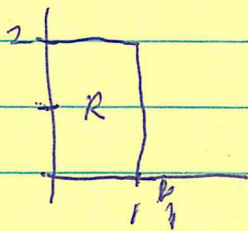
$$\partial_x(2e^x \cos(2y)) = 2e^x \cos(2y).$$

$$\text{Thus, } \oint_c F \cdot T \, ds = \iint_{\text{inside } c} 0 \, dA = \mathbf{0}.$$

Ex 2

Let $F = \langle xy, \frac{1}{3}x^3y + \frac{1}{2}x^2 \rangle$.

Let $R = [0, 1] \times [0, 2]$. Find the work done by F in going around ∂R ccw.



Sol

Let $M = xy$ and $N = \frac{1}{3}x^3y + \frac{1}{2}x^2$.

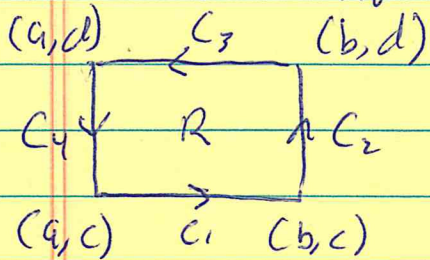
Then $M_y = x$ and $N_x = x^2y + x$.

Thus $N_x - M_y = x^2y$.

$$\oint_{\partial R} F \cdot T ds = \iint_R x^2y dA = \int_0^2 \int_0^1 x^2y dx dy = \frac{2}{3}.$$

We will do the proof of **Green's Theorem** where R is a rectangle of the form $[a, b] \times [c, d]$. We will discuss informally this can be used to prove the full version of **Green's Theorem**.

Green's Thm for a rectangle $R = [a, b] \times [c, d]$.



Let $C = C_1 \cup C_2 \cup C_3 \cup C_4 = \partial R$.

Let $F = \langle M, N \rangle$. We will

show that $\int_C M dx + N dy = \iint_R (N_x - M_y) dA$.

$$\text{Let } r_1(t) = \langle t, c \rangle \quad a \leq t \leq b,$$

$$r_2(t) = \langle b, t \rangle \quad c \leq t \leq d$$

$$r_3(t) = \langle -t, d \rangle \quad -b \leq t \leq -a$$

$$r_4(t) = \langle a, -t \rangle \quad -d \leq t \leq -c.$$

$$\oint_C M dx = \int_{C_1} M dx + \int_{C_2} M dx + \int_{C_3} M dx + \int_{C_4} M dx$$

$$\int_{C_1} M dx = \int_a^b M(t, c) \frac{dx}{dt} dt = \int_a^b M(t, c) dt = \int_a^b M(x, c) dx$$

$$\int_{C_2} M dx = \int_c^d M(b, t) \frac{dx}{dt} dt = \int_c^d 0 dt = 0.$$

$$\int_{C_3} M dx = \int_{-b}^{-a} M(-t, d) \frac{dx}{dt} dt = \int_{-b}^{-a} M(-t, d) (-dt) = \int_b^a M(s, d) ds$$

Let $s = -t$. Then $ds = -dt$ and $a \leq s \leq b$. Thus

$$\int_b^a M(s, d) ds = - \int_a^b M(s, d) ds = - \int_a^b M(x, d) dx.$$

$$\int_{C_4} M dx = \int_{-d}^{-c} M(a, -t) \frac{dx}{dt} dt = \int_{-d}^{-c} 0 dt = 0.$$

Thus $\int_C M dx = \int_a^b M(x, c) - M(x, d) dx$

Next consider, $\iint_R -M_y \, dA$. This equals

$$\int_a^b \int_c^d \frac{-2M}{2x} \, dy \, dx = - \int_a^b M(x,y) \Big|_{y=c}^{y=d} \, dx$$

$$= - \int_a^b M(x,d) - M(x,c) \, dx =$$

$$= \int_a^b M(x,c) - M(x,d) \, dx = \oint_c M \, dx.$$

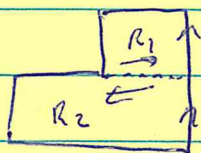
A similar calculation shows $\oint_c N \, dy = \iint_R \frac{2N}{2x} \, dA$.

Combining these gives **Green's Theorem** for rectangles like R .

$$\oint_c M \, dx + N \, dy = \iint_R N_x - M_y \, dA$$



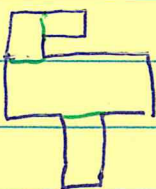
For the general case think about ...



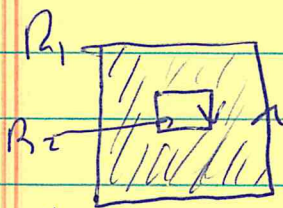
$$\text{let } R = R_1 \cup R_2.$$

$$\oint_{\partial R} F \cdot T ds = \oint_{\partial R_1} F \cdot T ds + \oint_{\partial R_2} F \cdot T ds.$$

Thus, Green's Theorem holds for region made from rectangles



We can cut out ~~region~~ rectangular holes.

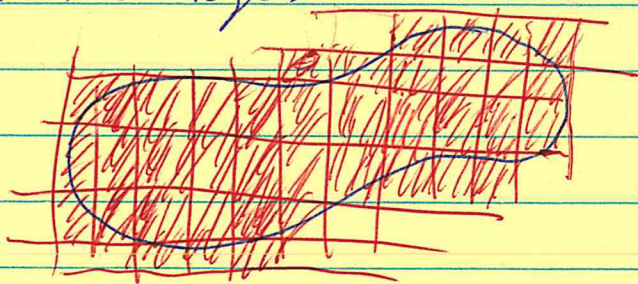


$$R = "R_1 - R_2"$$

$$\oint_{\partial R} F \cdot T ds = \oint_{\partial R_1} F \cdot T ds - \oint_{\partial R_2} F \cdot T ds = \oint_{\partial R_1} F \cdot T ds - \oint_{\partial R_2} F \cdot T ds.$$

$$= \iint_{R_1} -dA - \iint_{R_2} -dA = \iint_R -dA$$

For smooth regions, we can approximate using little rectangles.



Then take "limit" as the rectangles get smaller and smaller.

Ex Let $F(x,y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$. Let C_a be a circle

of radius a centered at the origin, oriented ccw.
Compute $\oint_{C_a} F \cdot dr$

Old Method Parametrize C_a with $r(t) = \langle a \cos t, a \sin t \rangle$
 $0 \leq t \leq 2\pi$.

$$F = \left\langle \frac{-a \sin t}{a^2}, \frac{a \cos t}{a^2} \right\rangle$$

$$r'(t) = \langle -a \sin t, a \cos t \rangle$$

$$\int_{C_a} F \cdot dr = \int_0^{2\pi} \left\langle \frac{-\sin t}{a}, \frac{\cos t}{a} \right\rangle \cdot \langle -a \sin t, a \cos t \rangle dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t dt = \int_0^{2\pi} 1 dt = 2\pi$$

Notice that the value of a did not matter.

New Method (Green's Thm) Let $M = \frac{-y}{x^2+y^2}$, $N = \frac{x}{x^2+y^2}$.

Now $F = \langle M, N \rangle$.

$$M_y = \frac{-(x^2+y^2) + y(2y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$N_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

Thus $M_y = N_x$.

$$\oint_{C_a} F \cdot dr = \iint_{\substack{\text{Disk} \\ \text{inside} \\ C_a}} 0 \, dA = 0.$$

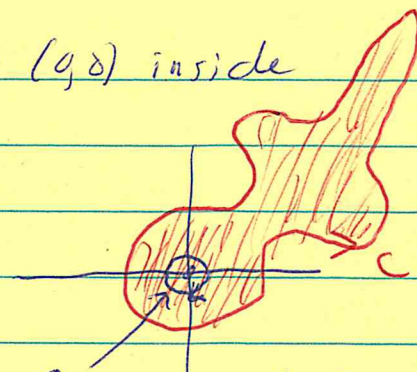
But $0 \neq 2\pi$!! What went wrong?

The problem is F is not defined at $(0,0)$
so Green's Thm does not apply.

But this "mistake" is useful!

Ex Let C be any ps scc that has $(0,0)$ inside the region R it bounds. Then

$$\oint_C F \cdot dr = 2\pi.$$



Pf Let $a > 0$ be small enough that C_a is inside R . Orient C ccw and C_a cw. Let R_1 be the region in between C and C_a . In R_1 , we can use Green's Thm.

$$\int_{C \cup C_a} F \cdot dr = 0 \quad \text{by Green's Thm.}$$

But

$$0 = \int_{C \cup C_a} F \cdot dr = \oint_C F \cdot dr + \oint_{C_a} F \cdot dr = \int_C F \cdot dr - 2\pi.$$

$$\text{Thus} \quad \oint_C F \cdot dr = 2\pi.$$

Area Consider a region R with boundary C parametrized by $r(t)$, ccw
The area inside is

$$\text{area} = \iint_R dA$$

If $F = \langle M, N \rangle$ is a vector field s.t. $N_x - M_y = 1$
then

$$\iint_R dA = \oint_C F \cdot T ds$$

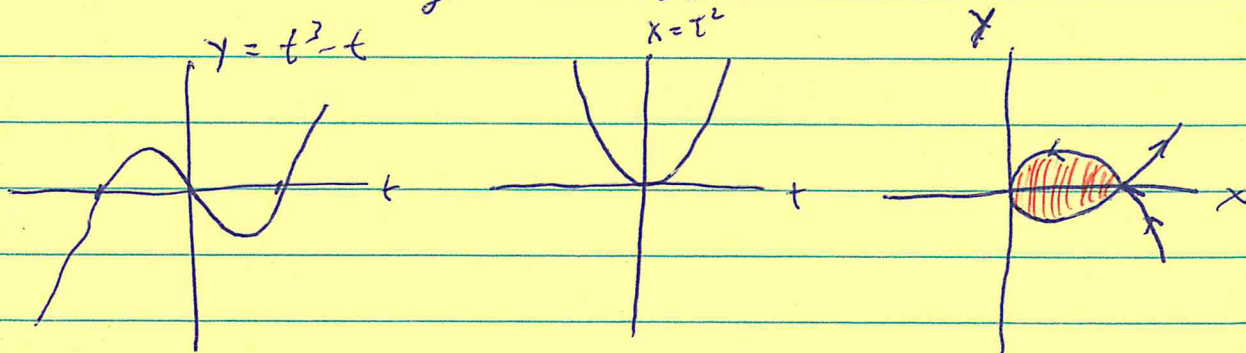
One such v.f. is $N = x, M = 0$. Thus

$$\begin{aligned} \text{area} &= \int_a^b \langle 0, x \rangle \cdot \langle x', y' \rangle dt \\ &= \int_a^b x(t) \frac{dy(t)}{dt} dt \end{aligned}$$

see also
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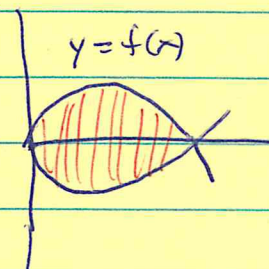
Ex Let $r(t) = \langle t^2, t^3 - t \rangle, -1 \leq t \leq 1$. Find the
area of the region enclosed.

Sol



$$A = \int_{-1}^1 t^2 (3t^2 - 1) dt = 2 \int_0^1 (3t^4 - t^2) dt = 2 \left(\frac{3}{5} - \frac{1}{3} \right) = \frac{8}{15}$$

Note: This example can also be done by conventional means. Since $x = t^2$, $t = \sqrt{x}$ and $y = x^{3/2} - x^{1/2}$ for $0 \leq x \leq 1$.



$$\text{Area} = 2 \int_0^1 x^{3/2} - x^{1/2} dx = 2 \left(\frac{2}{5} - \frac{2}{3} \right) = \frac{8}{15}$$

Ex Use the same region R but with density $\rho(x, y) = x^2 y^2$. Find the ~~center of mass~~ of mass.

Sol ~~$\bar{y} = 0$ by symmetry~~

$$\text{Mass} = \iint x^2 y^2 dA \quad \text{Find } F = \langle M, M \rangle \text{ s.t.}$$

$$N_x - M_y = x^2 y^2, \quad N = \frac{1}{3} x^3 y^2, \quad M = 0.$$

$$\iint x^2 y^2 dA = \int_{-1}^1 \left(\frac{1}{3} x^3 y^2 \right) \left(\frac{dy}{dt} \right) dt$$

$$= \frac{1}{3} \int_{-1}^1 t^6 (t^3 - t)^2 (3t^2 - 1) dt = \frac{64}{14305} \approx 0.0033$$

$$= \frac{2}{3} \int_0^1 3t^{14} - 7t^{12} + 5t^{10} - t^8 dt$$

$$= \frac{2}{3} \left(\frac{3}{15} - \frac{7}{13} + \frac{5}{11} - \frac{1}{9} \right) =$$