13.5 The Curl and Divergence of Vector Fields

Def Let $F = \langle P, Q, R \rangle$ be a vector field in $\mathbb{R}^3$. Then the divergence of F is the scalar function

$$\text{div } F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Also written as $\nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

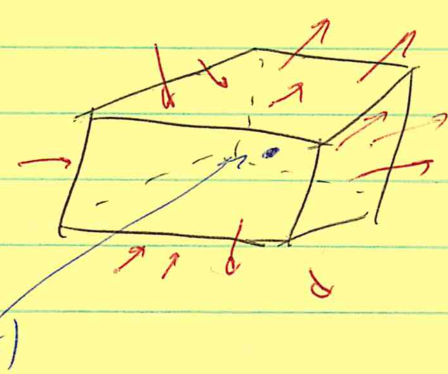
"del"

Ex $\nabla \cdot \langle x^3 y, z x e^y, 7x^2 \sin(z) \rangle$

$$= 3x^2 y + z x e^y + 7x^2 \cos(z).$$

Motivation Divergence measures how much a v.f. is "spreading" or "contracting" at each point. The definition used in physics books is in terms of the "local" flux (13.7):

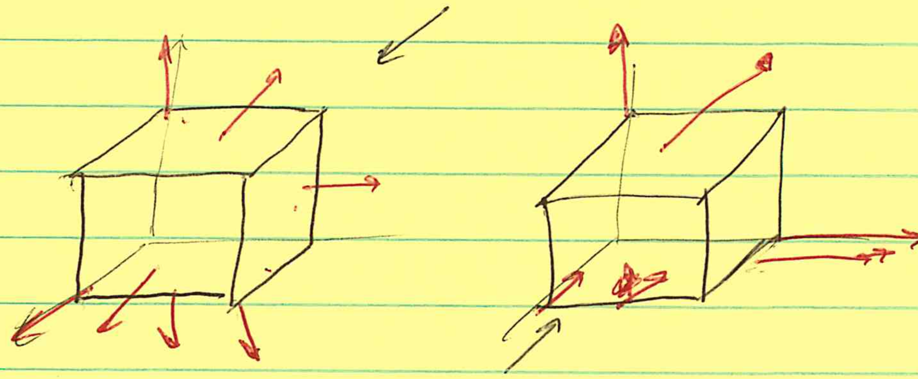
$$\text{div } F(x,y,z) = \lim_{\substack{\text{as cube} \\ \text{shrinks to} \\ \text{the point} \\ (x,y,z).}} \frac{\text{flux through boundary of small cube}}{\text{vol. of cube}}$$



It is then proven that this limits gives our formula

$$\text{div } F = P_x + Q_y + R_z.$$

Ex Let $F = \langle x, y, z \rangle$. Then $\nabla \cdot F = 1+1+1 = 3$, everywhere.



Ex Let $F = \langle -x, 2y, z \rangle$. Then $\operatorname{div} F = -1+2+1 = 2$.

Def Let $f(x,y,z)$ be a scalar function. Then the Laplacian of f is

$$\operatorname{div}(\operatorname{grad} f) = \nabla \cdot (\nabla f) = \nabla \cdot \langle f_x, f_y, f_z \rangle$$
$$= f_{xx} + f_{yy} + f_{zz}.$$

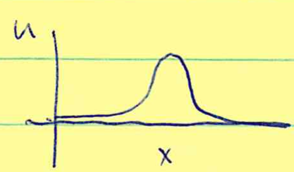
This is often written as $\nabla^2 f$.

Ex $\nabla^2(x^3y + z^2) = 6xy + 0 + 2 = 6xy + 2$

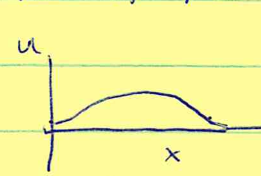
Application Let $u(x,y,z,t)$ be the temperature (or chem consentration) at $(x,y,z) \in \mathbb{R}^3$ and ~~time~~ at time t.

$$\frac{\partial u}{\partial t} = -k \nabla^2 u$$ is called the heat eq (or diffusion eq).

The idea is u drops faster if the concavity is sharper.



will spread fast



will spread slow

Def Let $F = \langle P, Q, R \rangle$ be a vector field on $\mathbb{R}^3$. Then

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

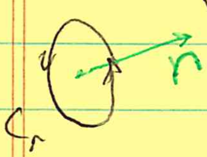
$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k$$

$$= \langle R_y - Q_z, \; P_z - R_x, \; Q_x - P_y \rangle.$$

Ex Let $F = \langle -cy, cx, 0 \rangle$. Then $\nabla \times F = \langle 0, 0, 2c \rangle$.

Motivation Curl measures "rotation" or "circulation" at each pt. Let C_r be an oriented circle of radius r, centered (x, y, z). The circulation around C_r is $\oint_{C_r} F \cdot T \, ds$.

Let n be a unit vector perpendicutular to the plane of C_r, pointing so that the RHR is following.



Then the curl of F is the unique vector field such that

$$n \cdot \operatorname{curl} F = \lim_{r \to 0} \frac{\oint_{C_r} F \cdot T \, ds}{\pi r^2}.$$

This is the "physics" definition of curl.

It can be shown it leads to our formula.

Green's Thm can be rewritten in terms of curl.

Suppose $F = \langle P, Q, 0 \rangle$. Then $\text{curl}\, F = \langle 0, 0, Q_x - P_y \rangle$.

If C_r is in the xy-plane, oriented ccw, then $n = \langle 0, 0, 1 \rangle$.

Thus, $n \cdot \text{curl}\, F = Q_x - P_y$.

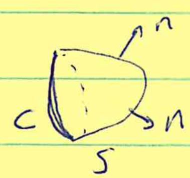
Green's Thm becomes

$$\oint_C F \cdot T\, ds = \iint_R Q_x - P_y \, dx\,dy = \iint_R n \cdot \text{curl}\, F \, dx\,dy.$$

C curve bounding R

In 13.8 we will see that for $F = \langle P, Q, R \rangle$

$$\oint_C F \cdot T\, ds = \iint_S (\text{curl}\, F) \cdot n \, dS.$$



This is called Stokes' Thm: It is the 3d version of Green's Thm.

Some Useful Properties

Thm 3, pg 790 $\quad \nabla \times \nabla f = 0$

Proof: $\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{xy} - f_{yx} \rangle$

$= \langle 0, 0, 0 \rangle.$

Thm 11, pg 792 $\quad \nabla \cdot (\nabla \times F) = 0$

Proof: $\nabla \cdot (\nabla \times \langle P, Q, R \rangle) =$

$\partial_x (R_y - Q_z) + \partial_y (P_z - R_x) + \partial_z (Q_x - P_y) =$

$R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$

From the exercises:

21. $\operatorname{div}(F + G) = \operatorname{div} F + \operatorname{div} G$
22. $\operatorname{curl}(F + G) = \operatorname{curl} F + \operatorname{curl} G$
23. $\operatorname{div}(fF) = \nabla f \cdot F + f \operatorname{div} F$ (Homework)
24. $\operatorname{curl}(fF) = \nabla f \times F + f \operatorname{curl} F$ (I'll do this one.)
25. $\operatorname{div}(F \times G) = (\operatorname{curl} F) \cdot G - F \cdot \operatorname{curl} G$

Not in book: $\operatorname{curl}(F \times G) = F(\nabla \cdot G) - G(\nabla \cdot F) + (G \cdot \nabla)F - (F \cdot \nabla)G$

Not in book: $\nabla(F \cdot G) = (F \cdot \nabla)G + (G \cdot \nabla)F + F \times (\operatorname{curl} G) + G \times (\operatorname{curl} F)$

26. $\operatorname{div}(\nabla f \times \nabla g) = 0.$ (Homework: Hint use #25 and Thm 3.)
27. $\operatorname{curl}(\operatorname{curl} F) = \operatorname{grad}(\operatorname{div} F) - \nabla^2 F.$

$\hookrightarrow \nabla^2 F = \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R \rangle.$

#24 curl$(fF) =$ f curl $F + \nabla f \times F$.

Pf Let $F = \langle P, Q, R \rangle$.

$$\nabla \times (fF) = \langle (fR)_y - (fQ)_z, (fP)_z - (fR)_x, (fQ)_x - (fP)_y \rangle$$

$$= \langle f_yR + fR_y - f_zQ - fQ_z, f_zP + fP_z - f_xR - fR_x, f_xQ + fQ_x - f_yP - fP_y \rangle$$

$$= \langle fR_y - fQ_z, fP_z - fR_x, fQ_x - fP_y \rangle + \langle f_yR - f_zQ, f_zP - f_xR, f_xQ - f_yP \rangle$$

$$= f\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = f(\nabla \times F). \checkmark$$

$$\begin{vmatrix} i & j & k \\ f_x & f_y & f_z \\ P & Q & R \end{vmatrix} = \langle f_yR - f_zQ, f_zP - f_xR, f_xQ - f_yP \rangle \checkmark$$

$$= \nabla f \times F.$$

Thus,

$$\nabla \times (fF) = f(\nabla \times F) + (\nabla f \times F) \quad \text{as claimed.}$$

#10 Discuss in class. For each expression below, does it make sense and if so is it a vector field or a scalar function? Assume f, g are scalar functions and F is a vector field.

(a) curl f — nonsense!

(b) grad f — vector field

(c) div F — scalar

(d) curl(grad f) — v. f.

(e) grad F — nonsense!

(f) grad(div F) — vector field

(g) div(grad f) — scalar

(h) grad(div f) — nonsense!

(i) curl curl F — v. f.

(j) div div F — nonsense!

(k) $\nabla f \times$ div F — nonsense!

(l) div(curl(grad f)) — scalar, in fact 0.