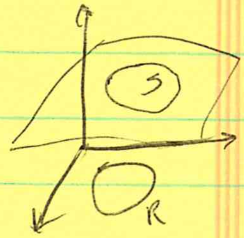
13.7

# Surface Integrals and Flux.

Recall, if $z = f(x,y)$ and $R$ is a region of the $xy$ plane, the surface area of the graph of $z = f(x,y)$ over $R$ is

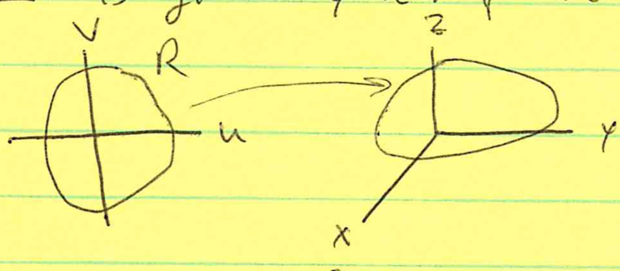



$$S.A. = \iint_S dS = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$

Let $g(x,y,z)$ be a "density" function on a ~~Surface~~ surface. Then the total "mass" (or charge) is,

$$\iint_R g(x, y, f(x,y)) \sqrt{1 + f_x^2 + f_y^2}\, dA$$

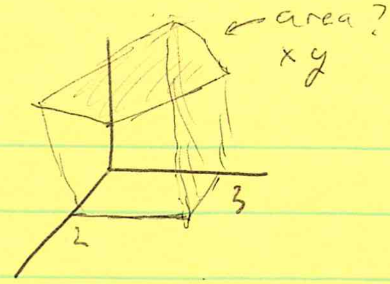
The text adds another twist to this. A <u>parametric surface</u> is given by a map: $r:(u,v) \to \langle x(u,v), y(u,v), z(u,v) \rangle$




Then $S.A. = \iint_R \| r_u \times r_v \| \, dA$ ← dudv

and

$$\text{"mass"} = \iint_R g(x(u,v), y(u,v), z(u,v)) \| r_u \times r_v \| \, dA$$




**Example** Let $f(x,y) = 3x+2y+1$
Let $R = [0,2]\times[0,3]$
Let $g(x,y,z) = xyz$.
Find $\iint_R g\,dS$.

**Solution**

$$\int_0^3\int_0^2 g(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}\,dx\,dy$$

$$= \int_0^3\int_0^2 xy(3x+2y+1)\sqrt{1+9+4}\,dx\,dy$$

$$= \sqrt{14}\int_0^3\int_0^2 3x^2y+2xy^2+xy\,dx\,dy$$

$$= \sqrt{14}\int_0^3 8y+4y^2+2y\,dy$$

$$= \sqrt{14}\left(\frac{10\cdot 9}{2}+\frac{4\cdot 27}{3}\right) = \sqrt{14}\,(45+36)$$

$$= \boxed{81\sqrt{14}}$$

center $(0,0,0)$

**Example** Let the charge density of a metal spherical surface of radius $R$ be proportional to the distance to the xy-plane. Find the total charge.

**Solution** Let $g(x,y,z) = k|z| = kR|\cos\phi|$.

Trick: $dS = R^2 \sin\phi \, d\phi \, d\theta$ [discuss this]

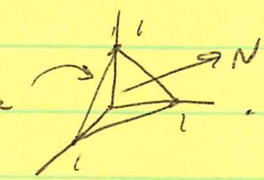
$$Q = \int_0^{2\pi}\int_0^{\pi} kR|\cos\phi| \, R^2 \sin\phi \, d\phi \, d\theta$$

$$= 4\pi k R^3 \int_0^{\pi/2} \cos\phi \sin\phi \, d\phi$$

$u = \sin\phi \quad du = \cos\phi \, d\phi$

$$= 4\pi k R^3 \int_0^1 u \, du = \boxed{2\pi k R^3 = \frac{kR}{2}(4\pi R^2)}$$

# Flux

**Ex** Let $F = \langle 3x, -y, z \rangle$. Let $S$ be .

Find the flux of $F$ out (up) through $S$.

**Solution:**

$$\text{Flux} = \iint_S F \cdot N \, dS \qquad [\text{Explain}]$$

Find $N$: Eq of plane is $x + y + z = 1$.
So $N = \langle 1, 1, 1 \rangle / \sqrt{3}$.

$$F \cdot N = (3x - y + z)/\sqrt{3}$$

Find $dS$. Rewrite eq of plane as $z = f(x,y) = 1 - x - y$.

$$dS = \sqrt{1 + (-1)^2 + (-1)^2} \, dx\,dy = \sqrt{3} \, dx\,dy.$$

Thus,

$$\text{Flux} = \int_0^1 \int_0^{1-y} \frac{(3x - y + \underbrace{z}_{=1-x-y})}{\cancel{\sqrt{3}}} \cancel{\sqrt{3}} \, dx\,dy$$

$$= \int_0^1 \int_0^{1-y} 1 + 2x - 2y \, dx\,dy$$

$$= \int_0^1 x + x^2 - 2yx \Big|_0^{1-y} dy$$

$$= \int_0^1 (1-y) + (1-y)^2 - 2y(1-y) \, dy$$

$$= \int_0^1 1 - y + 1 - 2y + y^2 - 2y + 2y^2 \, dy$$

$$= \int_0^1 2 - 5y + 3y^2 \, dy = 2 - \frac{5}{2} + 1 = \boxed{\frac{1}{2}}$$

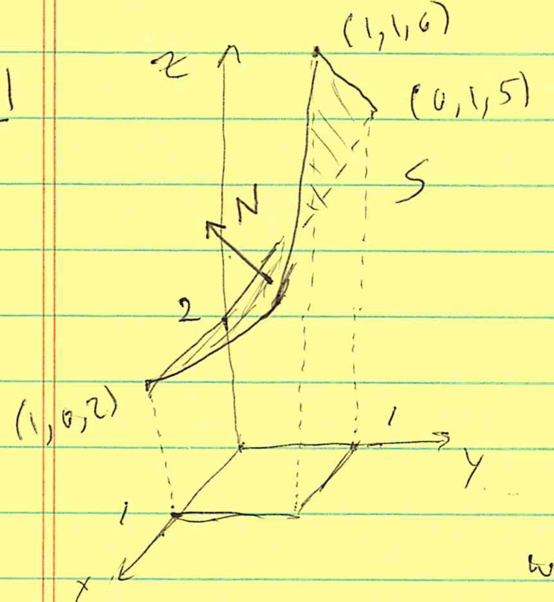
**Ex** Let $f(x,y) = x^2y^2 + 3y + 2$.

Let $R = [0,1]\times[0,1]$ and let $S$ be the graph of $z = f(x,y)$ over the region $R$.

Let $F(x,y,z) = \langle x^2, y, z \rangle$.

Find the flux of $F$ up through $S$.

**Sol**




Here is $S$, the graph of $z = f(x,y)$.

To find $N$ let $g(x,y,z) = f(x,y) - z$.

Then $S$ is a level surface of $g$, and

$$N = \pm \frac{\nabla g}{|\nabla g|}$$

Since $g(x,y,z) = x^2y^2 + 3y + 2 - z$,

we have $\nabla g = \langle 2xy^2, 2x^2y + 3, -1 \rangle$.

To get $N$ pointing upward we use the $(-)$ sign.

$$N = \frac{-\langle 2xy^2, 2x^2y+3, -1\rangle}{|\nabla g|}.$$

Next,

$$F\cdot N = (-2x^3y^2 - 2x^2y^2 - 3y + z)/|\nabla g|$$

$\hookrightarrow z = x^2y^2 + 3y + 2$

$$= (2 - 2x^3y^2 - x^2y^2)/|\nabla g|.$$

Now, $dS = \underbrace{\sqrt{1+(f_x)^2+(f_y)^2}\,dx\,dy = |\nabla g|}_{\text{check this!}}\,dx\,dy$

Thus,

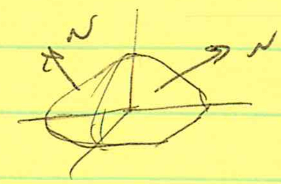
$$\text{flux} = \int_0^1\int_0^1 \frac{2-2x^3y^2-x^2y^2}{\cancel{|\nabla g|}}\,\cancel{|\nabla g|}\,dx\,dy$$

$$= \int_0^1 2 - \frac{y^2}{2} - \frac{y^2}{3}\,dy = \int_0^1 2 - \frac{5y^2}{6}\,dy$$

$$= 2 - \frac{5}{6}\cdot\frac{1}{3} = \frac{36}{18} - \frac{5}{18} = \frac{31}{18} = 1.7\overline{2}$$

<u>Example</u> Let $F = \langle 3x, -y, z\rangle$; Let $S$ be upper half of the unit sphere, $x^2+y^2+z^2=1$.

Find the flux of $F$ up through $S$.




<u>Solution</u> $N = \langle x, y, z\rangle \cancel{/} \|\langle x,y,z\rangle\| = \langle x,y,z\rangle$.

$F \cdot N = 3x^2 - y^2 + z^2 \qquad \rho = 1.$

$= 3\cos^2\theta \sin^2\phi - \sin^2\theta \sin^2\phi + \cos^2\phi$

$dS = 1^2 \sin\phi \, d\phi \, d\theta$

$$\text{Flux} = \int_0^{2\pi}\int_0^{\pi/2} [3\cos^2\theta - \sin^2\theta]\sin^3\phi + \cos^2\phi\sin\phi \, d\phi \, d\theta$$

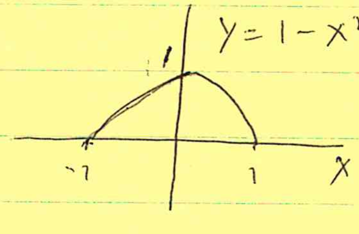
$$= \int_0^{\pi/2} (3\pi - \pi)\sin^3\phi + 2\pi\cos^2\phi\sin\phi \, d\phi$$

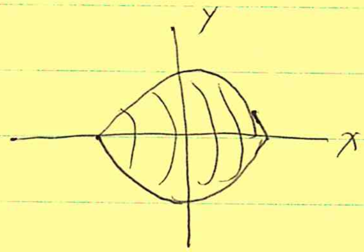
$$= 2\pi \int_0^{\pi/2} (1-\cos^2\phi)\sin\phi + \cos^2\phi\sin\phi \, d\phi$$

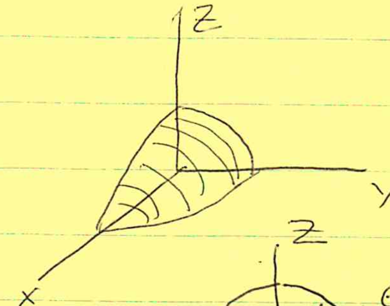
$$= 2\pi \int_0^{\pi/2} \sin\phi \, d\phi = \boxed{2\pi}$$

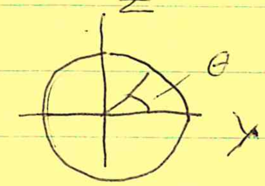
<u>Ex</u> Let $F = \langle x, y, z \rangle$. Let $S$ be the surface formed by rotating the curve $y = 1 - x^2$, $-1 \le x \le 1$, about the x-axis. Find the flux of $F$ out through $S$.

<u>Sol</u>.




$$\text{Flux} = \iint_S F \cdot N \, dS.$$

We parametrize the surface $S$ as

$$r(x, \theta) = \langle x, (1-x^2)\cos\theta, (1-x^2)\sin\theta \rangle$$

Then $dS = |r_x \times r_\theta| \, dx \, d\theta$, $N = \pm \dfrac{r_x \times r_\theta}{|r_x \times r_\theta|}$.

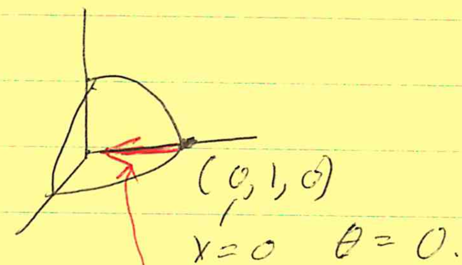
$$r_x = \langle 1, -2x\cos\theta, -2x\sin\theta \rangle$$
$$r_\theta = \langle 0, -(1-x^2)\sin\theta, (1-x^2)\cos\theta \rangle$$

$$r_x \times r_\theta = \langle -2x(1-x^2)\cos^2\theta - 2x(1-x^2)\sin^2\theta, -(1-x^2)\cos\theta, -(1-x^2)\sin\theta \rangle$$

$$= \langle -2x(1-x^2), -(1-x^2)\cos\theta, -(1-x^2)\sin\theta \rangle$$

$$= -(1-x^2)\langle 2x, \cos\theta, \sin\theta \rangle.$$

It turns out $\frac{r_x \times r_\theta}{|r_x \times r_\theta|}$ point inward. You can see this by checking one pt on $S$. The point $(0,1,0)$ corresponds to $x=0$, $\theta=0$.




$$(r_x \times r_\theta)(0,0) = -\langle 0, 1, 0\rangle$$

So, we need to use $N = -\frac{r_x \times r_\theta}{|r_x \times r_\theta|}$

$$= (1-x^2)\langle 2x, \cos\theta, \sin\theta\rangle \big/ |r_x \times r_\theta|.$$

$$F \cdot N = \langle x, \underbrace{y}_{(1-x^2)\cos\theta}, \underbrace{z}_{(1-x^2)\sin\theta}\rangle \cdot \big((1-x^2)\langle 2x, \cos\theta, \sin\theta\rangle\big) \big/ |r_x \times r_\theta|$$

$$= (1-x^2)\big(2x^2 + (1-x^2)\cos^2\theta + (1-x^2)\sin^2\theta\big) \big/ |r_x \times r_\theta|$$

$$= (1-x^2)\big(2x^2 + (1-x^2)\big) \big/ |\quad|$$

$$= (1-x^2)(1+x^2) \big/ |\quad|$$

$$= \frac{1-x^4}{|r_x \times r_\theta|}$$

$$\text{Flux} = \iint_S F \cdot N \, dS =$$

$$\int_0^{2\pi}\int_{-1}^{1} \frac{1 - x^4}{|r_x \times r_\theta|} |r_x \times r_\theta| \, dx \, d\theta$$

$$= \int_0^{2\pi}\int_{-1}^{1} 1 - x^4 \, dx \, d\theta$$

$$= 2\pi \cdot 2 \int_0^1 1 - x^4 \, dx$$

$$= 4\pi\left(1 - \frac{1}{5}\right) = \frac{16\pi}{5}$$