

13.8

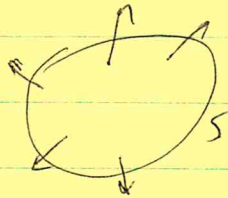
Stokes' Thm (or the curl theorem)Thm

Let  $S$  be a piecewise smooth surface in  $\mathbb{R}^3$  with boundary. Assume  $S$  is oriented and that the direction of  $C$  follows the RHR. Let  $F$  be a vector field that has continuous partial derivatives on an open region containing  $S$ . Then

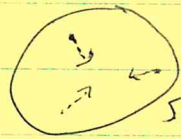
$$\oint_C F \cdot T ds = \iint_S (\text{curl } F) \cdot n dS$$

Remarks

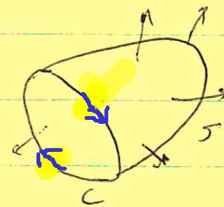
① A surface is oriented by choosing a consistent normal direction at each interior pt.



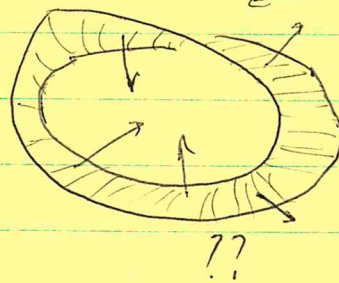
outward orientation



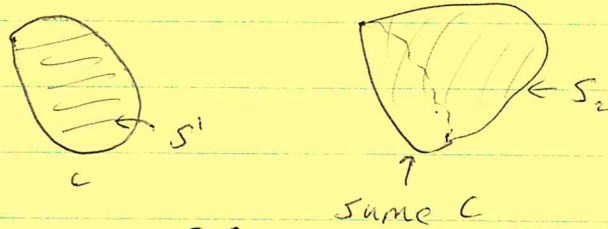
inward.



Some surfaces do not allow this. They are said to be non-orientable. The famous Möbius band is an example  
 ← half twist.



- ② Stokes' Thm implies the surface integral of  $(\text{curl } F) \cdot n$  depends only on the boundary curve (or curves). So we have "surface independence".



$$\iint_{S_1} (\text{curl } F) \cdot n \, dS = \iint_{S_2} (\text{curl } F) \cdot n \, dS$$

- ③ If  $C$  and  $S$  are in  $\mathbb{R}^2$  then Stokes' Thm becomes Green's Thm.

Let  $F = \langle P, Q, R \rangle$ . If  $S \subset \mathbb{R}^2$  use  $n = \langle 0, 0, 1 \rangle$ .  $C$  ccw

$$\begin{aligned} \text{Then } \oint_C F \cdot T \, ds &= \iint_S (\text{curl } F) \cdot \langle 0, 0, 1 \rangle \, dS \\ &= \iint_S Q_y - P_x \, dx \, dy. \end{aligned}$$

$dA = dx \, dy$

- ④  $\nabla \times F = 0 \Rightarrow \oint_C F \cdot T \, ds = 0$  for any  $C$ . (Just find a surface, orientable, with boundary  $C$ .)

This is path independence. We already know

Path indep.  $\Rightarrow F = \nabla f$  for some  $f$ .

and  $\nabla \times (\nabla f) = 0 \Rightarrow \nabla \times F = 0$ .

So,

$\nabla \times F = 0 \iff$  Path independence

Ex 1  $F(x, y, z) = \langle xy, yz, xz \rangle$ .

$S =$  unit disk in  $xy$ -plane  $\{(x, y, 0) \mid x^2 + y^2 \leq 1\}$ .

$C = \partial S =$  unit circle  $\{(x, y, 0) \mid x^2 + y^2 = 1\}$ .

Find  $\oint_C F \cdot T \, ds$ .

Sol  $\oint_C F \cdot T \, ds = \iint_S (\nabla \times F) \cdot n \, dS$ .

$$n = \langle 0, 0, 1 \rangle, \quad \nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy & yz & xz \end{vmatrix} = \langle -y, -z, -x \rangle.$$

Thus,  $(\nabla \times F) \cdot n = -x$ .

We use polar coordinates.  $dS = r \, dr \, d\theta$

$$\int_0^{2\pi} \int_0^1 -(r \cos \theta) r \, dr \, d\theta = 0.$$

Ex 2 Let  $S$  be the graph of  $z = 9 - x^2 - y^2$  for  $z \geq 0$ .  
Let  $C = \partial S : x^2 + y^2 = 9, z = 0$ .  
Let  $F = \langle 3z, 4x, 2y \rangle$ .

Find  $\oint_C F \cdot T ds$  3 different ways!

Sol I (old way). Let  $r(t) = \langle 3 \cos t, 3 \sin t, 0 \rangle, 0 \leq t \leq 2\pi$ .

$$\oint_C F \cdot T ds = \int_0^{2\pi} F \cdot \frac{dr}{dt} dt$$

$$\frac{dr}{dt} = \langle -3 \sin t, 3 \cos t, 0 \rangle$$

$$F = \langle 3 \cdot 0, 4 \cdot 3 \cos t, 2 \cdot 3 \sin t \rangle \\ = \langle 0, 12 \cos t, 6 \sin t \rangle$$

$$F \cdot \frac{dr}{dt} = 36 \cos^2 t$$

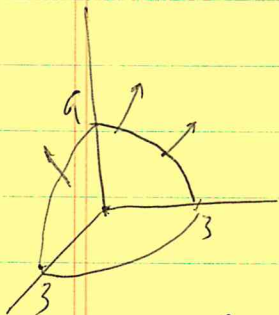
$$\int_0^{2\pi} 36 \cos^2 t dt = \boxed{36\pi}$$

Sol II By Stokes' Thm  $\oint_C F \cdot T ds = \iint_S (\nabla \times F) \cdot n dS$ .

$$\nabla \times F = \langle 2, 3, 4 \rangle$$

Find  $n$ . Let  $g(x, y, z) = z + x^2 + y^2$ . Then  $g = 9$  gives our surface.

$$\nabla g = \langle 2x, 2y, 1 \rangle \quad n = \frac{\pm \nabla g}{|\nabla g|}$$



From picture use  $+$ .

$$dS = |\nabla g| dA$$

$$\iint (\nabla \times F) \cdot \frac{\nabla g}{|\nabla g|} |\nabla g| dA = \iint \langle 2, 3, 4 \rangle \cdot \langle 2x, 2y, 1 \rangle dA$$

$$= \iint 4x + 6y + 4 dA = \int_0^{2\pi} \int_0^3 (4r \cos \theta + 6r \sin \theta + 4) r dr d\theta$$

$$= 8\pi \int_0^3 r dr = 8\pi \cdot \frac{9}{2} = 36\pi$$

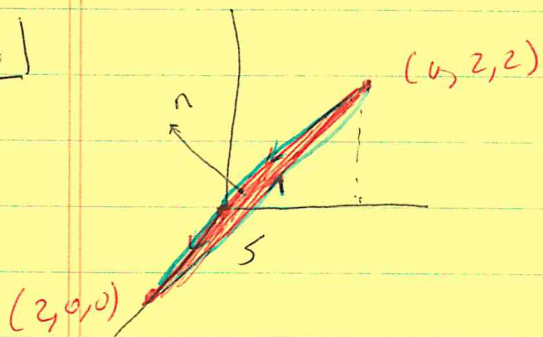
Sol III Replace  $S$  with the disk of radius 3 in the  $xy$ -plane, center  $(0, 0, 0)$ . Now  $n = \langle 0, 0, 1 \rangle$ .

$$\langle \nabla \times F \rangle \cdot n = \langle 2, 3, 4 \rangle \cdot \langle 0, 0, 1 \rangle = 4$$

$$\int_0^{2\pi} \int_0^3 4 r dr d\theta = 36\pi$$

Ex 3 Let  $C$  be the boundary of the triangle with vertices  $(0,0,0)$ ,  $(2,0,0)$ ,  $(0,2,2)$  oriented ccw when viewed from above. Let  $F = \langle y, z, x \rangle$ . Find  $\oint_C F \cdot T ds$ .

Sol



By Stokes' Thm  $\oint_C F \cdot T ds = \iint_S (\text{curl } F) \cdot n ds$ .

$$\nabla \times F = \langle -1, -1, -1 \rangle.$$

Find  $n$ .  $n = \frac{\pm \langle 2, 0, 0 \rangle \times \langle 0, 2, 2 \rangle}{1} = \frac{\pm \langle 0, -4, 4 \rangle}{\sqrt{32}}$

Use  $n = \langle 0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ .

$$(\nabla \times F) \cdot n = \langle -1, -1, -1 \rangle \cdot \langle 0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = 0.$$

Hence  $\oint_C F \cdot T ds = \iint_S 0 ds = 0$ .

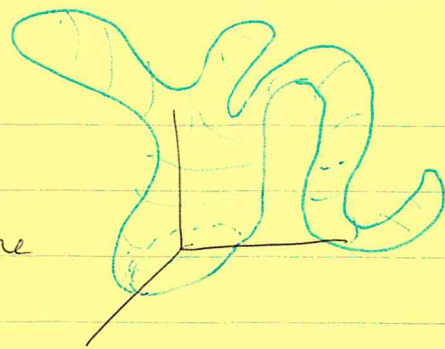
What if

we had gotten  $(\nabla \times F) \cdot n = 7$ . Then the answer would be  $7 \times$  the area of the triangle  $= 7 \cdot 2\sqrt{2} = 14\sqrt{2}$ .

Ex 4

Let  $F = \langle 3y, -2x, xyz \rangle$ .

Let  $S$  be as in the figure  
with  $\partial S = C =$  unit circle in  $xy$ -plane



$$\text{Find } \iint_S (\nabla \times F) \cdot n \, dS.$$

Sol

By Stokes' Thm we can replace  $S$  with the unit disk in the  $xy$ -plane

$$\iint_{\text{disk}} (\nabla \times F) \cdot \langle 0, 0, 1 \rangle \, dA$$

$$\nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & xyz \end{vmatrix} = \langle \text{don't care}, \text{don't care}, -5 \rangle$$

$$\iint_{\text{unit disk}} -5 \, dA = -5\pi.$$

Ex 5 Let  $S$  and  $C = \partial S$  satisfy the conditions of Stokes' Thm. Let  $f$  and  $g$  be scalar functions with all second order ~~partial~~ partial derivatives continuous. Prove that

$$\oint_C (f \nabla g) \cdot T \, ds = \iint_S (\nabla f \times \nabla g) \cdot n \, dS$$

Proof By Stokes' Thm

$$\oint_C (f \nabla g) \cdot T \, ds = \iint_S \nabla \times (f \nabla g) \cdot n \, dS.$$

But  $\nabla \times (f \nabla g) = f(\nabla \times \nabla g) + \nabla f \times \nabla g$  by #26 in 13.5.

And  $\nabla \times \nabla g = \mathbf{0}$  by Thm 3 in 13.5.

Thus,

$$\oint_C (f \nabla g) \cdot T \, ds = \iint_S (\nabla f \times \nabla g) \cdot n \, dS$$

as claimed.

## Concluding Remarks

We have  $\nabla \times F = 0 \Leftrightarrow F = \nabla f$  for some  $f$   
 $\Leftrightarrow$  path independence

It turns out  $\nabla \cdot F = 0 \Leftrightarrow F = \nabla \times G$  for some vector field  $G$ .

If  $F = \nabla f$ , then  $f$  is called a potential function for  $F$ .

If  $F = \nabla \times G$ , then  $G$  is called a vector potential for  $F$ .  
(This works for magnetic fields.)