13.4 **The Divergence Thm** Let V be a connected region of $\mathbb{R}^3$ with boundary S, a piecewise smooth closed surface. Let F be a vector field with continuous partial derivatives on an open region containing V. Then

$$\text{Flux out of } \vec{F} \text{ through } S = \iint_S F \cdot N \, dS = \iiint_V \nabla \cdot F \, dV.$$

Ex 1 Let $F = \langle 2x, z, -y \rangle$. Let S be the unit sphere: $x^2+y^2+z^2=1$. Find the flux of F out through S.

Sol I (Old Way). Use $N = \langle x, y, z \rangle$. $F \cdot N = 2x^2 + yz \cancel{-} yz = 2x^2$.

$dS = R^2 \sin\phi \, d\theta \, d\phi = \sin\phi \, d\theta \, d\phi$ since $R=1$.

$$\text{Flux} = \int_0^{\pi}\int_0^{2\pi} \underbrace{2\cos^2\theta \sin^2\phi}_{= 2x^2} \sin\phi \, d\theta \, d\phi$$

$$= 2\pi \int_0^{\pi} \sin^3\phi \, d\phi = 2\pi \int_0^{\pi} (1-\cos^2\phi)\sin\phi \, d\phi$$

Let $u = \cos\phi$. Then $du = -\sin\phi \, d\phi$

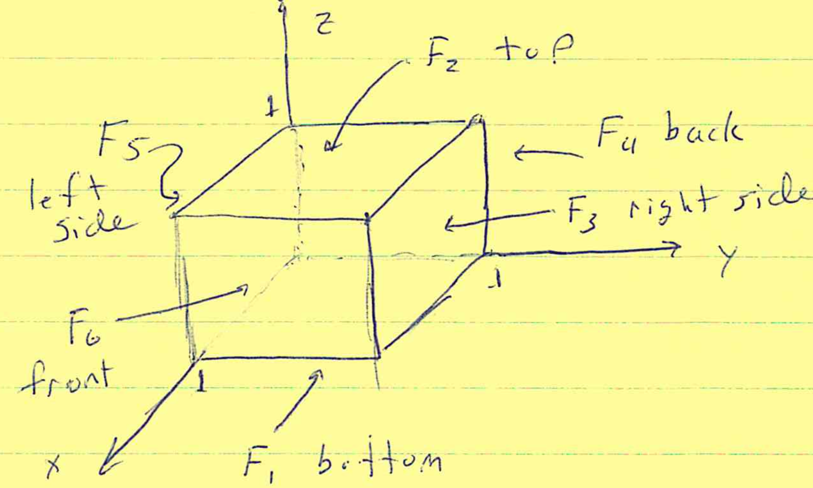
$$= -2\pi \int_1^{-1} 1-u^2 \, du = 2\pi \int_{-1}^{1} 1-u^2 \, du = 4\pi \int_0^1 1-u^2 \, du$$

$$= 4\pi \left(u - \frac{u^3}{3}\right)\Big|_0^1 = \frac{8\pi}{3}.$$

Sol II $\nabla \cdot F = 2+0+0$.

$\text{Flux} = \iint F \cdot N \, dS = \iiint 2 \, dV = 2\left(\frac{4}{3}\pi R^3\right) = \frac{8\pi}{3}$, since $R=1$.

Proof of the Divergence Theorem for the cube $V = [0,1]^3$.



Let $F = \langle P, Q, R \rangle$.

Boundary of the cube $V = F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5 \cup F_6$.

For each F_i there is a unit normal vector $n_i = \langle \alpha_{i1}, \alpha_{i2}, \alpha_{i3} \rangle$.

$n_1 = \langle 0, 0, -1 \rangle$

$n_2 = \langle 0, 0, 1 \rangle$

$n_3 = \langle 0, 1, 0 \rangle$

$n_4 = \langle -1, 0, 0 \rangle$

$n_5 = \langle 0, -1, 0 \rangle$

$n_6 = \langle 1, 0, 0 \rangle$.

$\nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. $F \cdot n_i = P\alpha_{i1} + Q\alpha_{i2} + R\alpha_{i3}$.

I claim that

$$(\ast)\quad \sum_{i=1}^{6} \iint_{F_i} P\,\alpha_{i1}\, dS = \iiint_V \frac{\partial P}{\partial x}\, dV,$$

$$\sum_{i=1}^{6} \iint_{F_i} Q\,\alpha_{i2}\, dS = \iiint_V \frac{\partial Q}{\partial y}\, dV, \quad \text{and}$$

$$\sum_{i=1}^{6} \iint_{F_i} R\,\alpha_{i3}\, dS = \iiint_V \frac{\partial R}{\partial z}\, dV.$$

Combining these give $\displaystyle\iint_{\substack{\text{surface}\\ \text{of cube}}} F \cdot n\, dS = \iiint_{\text{cube}} \nabla \cdot F\, dV.$

I'll only show $(\ast)$ as the other two are done similarly.

$$\sum_{i=1}^{6} \iint_{F_i} P\,\alpha_{i1}\, dS = \iint_{F_4} -P\, dS + \iint_{F_6} P\, dS$$

$$= -\int_0^1\int_0^1 P(0,y,z)\, dy\, dz + \int_0^1\int_0^1 P(1,y,z)\, dy\, dz$$

$$= \int_0^1\int_0^1 P(1,y,z) - P(0,y,z)\, dy\, dz.$$

But, by the Fund. Thm of Calculus,

$$\int_0^1\int_0^1\int_0^1 \frac{\partial P}{\partial x}\, dx\, dy\, dz = \int_0^1\int_0^1 P(1,y,z) - P(0,y,z)\, dy\, dz.$$

Done!

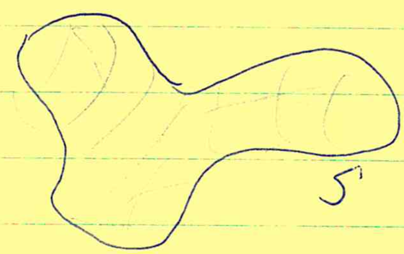
It is easy to extend this proof to any rectanular region with sides parallel to the coordinates planes.

It is not too hard to extend it to regions built up from such blocks.

The general proof of the Divergence Thm involves covering the region with smaller and smaller cubes and taking the limit as cube volume $\to 0$.

Ex 2 Let $F = \langle x, y, z \rangle$. Let S be a surface that is the boundary of a region V which has volume 5 units. Find the flux of F out through S.

Sol.

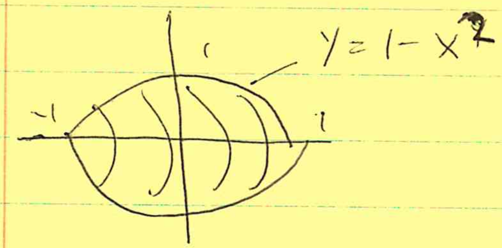


$$\iint F \cdot N \, dS = \iiint \nabla \cdot F \, dV = \iiint 3 \, dV$$

$$= 3 \cdot \iiint dV = 3 \cdot 5 = 15.$$

Ex (This is the same as the last example from the 13.7 lecture.) Let $F = \langle x, y, z \rangle$. Let S be the surface formed by rotating the curve $y = 1 - x^2$, $-1 \le x \le 1$, about the x-axis. Find the flux of F out through S.

Sol



$$\text{Flux} = \iint_S F \cdot N \, dS = \iiint_V \nabla \cdot F \, dV$$ by the divergence thm. $\nabla \cdot F = 1 + 1 + 1 = 3$. So

$$\text{Flux} = 3 \cdot (\text{vol inside } S).$$

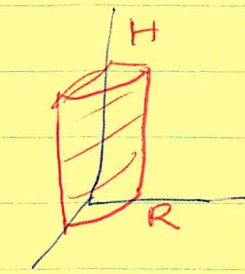
We use the disk method.

$$V = \int_{-1}^{1} \pi \left[1 - x^2\right]^2 dx = 2\pi \int_0^1 x^4 - 2x + 1 \, dx$$

$$= 2\pi\left(\frac{1}{5} - \frac{2}{3} + 1\right) = \frac{16\pi}{15}$$

Thus, $\text{Flux} = 3\left(\frac{16\pi}{15}\right) = \frac{16\pi}{5}$.

Ex Let V be the cylinder with radius R and height H shown.
Let $F = \langle x^3, xz^2, xyz \rangle$.
Find the flux out through the boundary of V.



Sol 1

$$\text{Flux} = \iint_{\partial V} F \cdot N \, dS = \iiint_V \nabla \cdot F \, dV$$

$$= \int_0^H \int_0^{2\pi} \int_0^R 3x^2 + 0 + xy \quad r\,dr\,d\theta\,dz$$

$$= \int_0^H \int_0^{2\pi} \int_0^R 3r^3\cos^2\theta + r^3\cos\theta\sin\theta \, dr\,d\theta\,dz$$

$$= H \int_0^{2\pi} \frac{3}{4}R^4\cos^2\theta + \frac{R^4}{4}\cos\theta\sin\theta \, d\theta$$

$$= H \cdot \frac{3}{4}R^4 \cdot \pi = \frac{3\pi H R^4}{4}.$$

Sol 2 Just for fun we will do it the old way!

$$\text{Flux} = \iint_{\partial V} F \cdot N \, dS = \iint_{top} F \cdot N \, dS + \iint_{side} F \cdot N \, dS + \iint_{bottom} F \cdot N \, dS$$

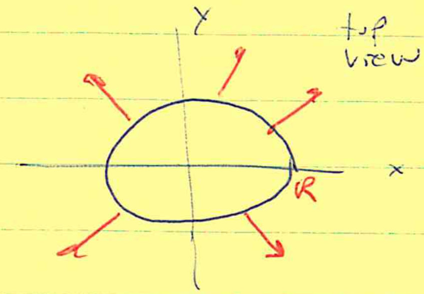
Top: $N = \langle 0, 0, 1 \rangle$. $F \cdot N = xyz$, but $z = H$.

$$\iint_{top} F \cdot N \, dS = H \int_0^{2\pi} \int_0^R r^2\cos\theta\sin\theta \; r\,dr\,d\theta = 0$$

Bottom: $N = \langle 0, 0, -1 \rangle$. $F \cdot N = -xyz$, but $z = 0$.

$$\iint_{\text{bottom}} F \cdot N \, dS = \iint 0 \, dS = 0.$$

Side $N = \langle x, y, 0 \rangle / \sqrt{x^2+y^2} = \left\langle \frac{x}{R}, \frac{y}{R}, 0 \right\rangle$.



$$F \cdot N = \frac{x^4}{R} + \frac{xyz^2}{R}$$

$$\iint_{\text{side}} F \cdot N \, (R \, d\theta \, dz)$$

$$= \int_0^H \int_0^{2\pi} R^4 \cos^4\theta + \overset{0}{\cancel{R^2 \cos\theta \sin\theta \, z^2}} \, d\theta \, dz$$

$$= HR^4 \int_0^{2\pi} \cos^4\theta \, d\theta$$

You can check $\int_0^{2\pi} \cos^4\theta \, d\theta = \frac{3\pi}{4}$. (see pg 319)

So, the answer is $\frac{3\pi H R^4}{4}$

But, we did learn something new. There is no net flux out of the top and no flux at all on the bottom.

Ex Let $F = \dfrac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$.

Let V be a bounded connected region of $\mathbb{R}^3$ with smooth boundary S and suppose $(0,0,0)$ is inside V. Find the flux of F out through S.

Sol) Step 1 Show $\nabla \cdot F = 0$ everywhere except the origin where it is undefined.

$$\nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \left\langle \frac{x}{(\text{—})^{3/2}}, \frac{y}{(\text{—})^{3/2}}, \frac{z}{(\text{—})^{3/2}}\right\rangle$$

$$= \frac{1\cdot(\quad)^{3/2} - \frac{3}{2}x(\quad)^{\frac{1}{2}}2x}{(\text{—})^3} + \frac{1\cdot(\text{—})^{3/2} - \frac{3}{2}y(\quad)^{\frac{1}{2}}2y}{(\quad)^3} + \frac{1\cdot(\text{—})^{3/2} - \frac{3}{2}z(\quad)^{\frac{1}{2}}2z}{(\quad)^3}$$

$$= \frac{3(\quad)^{3/2} - 3x^2(\quad)^{\frac{1}{2}} - 3y^2(\quad)^{\frac{1}{2}} - 3z^2(\quad)^{\frac{1}{2}}}{(\quad)^3}$$

$$= \frac{3(\quad)^{3/2} - 3(x^2+y^2+z^2)(\quad)^{\frac{1}{2}}}{(\quad)^3}$$

$$= \frac{3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3} = 0$$

except at $(0,0,0)$ where it is undefined.

Step 2 Let B be a ball of radius a, center $(0,0,0)$ with B inside V. Let $S_1 = \partial B$. Let $V_1 = V -$ inside of B. Then the divergence thm can be applied to V_1.

$$\iiint_{V_1} \nabla \cdot F = \iint_S F \cdot N \, dS - \iint_{S_1} F \cdot N \, dS$$

But $\nabla \cdot F = 0$ inside V_1. Thus,

$$\iint_S F \cdot N \, ds = \iint_{S_1} F \cdot N \, dS.$$

Step 3 Find flux out of the ball through S_1.

$$\iint_{S_1} F \cdot N \, dS = \iint_{S_1} \frac{\langle x, y, z\rangle}{\underset{\overset{\shortparallel}{a^3}}{(x^2+y^2+z^2)^{3/2}}} \cdot \frac{\langle x, y, z\rangle}{a} \, dS$$

$$= \frac{1}{a^4} \iint_{S_1} \underbrace{x^2+y^2+z^2}_{=a^2} \, dS = \frac{1}{a^2} \iint_{S_1} dS = \frac{1}{a^2}(4\pi a^2)$$

$$= 4\pi.$$

Thus, the flux $\iint_S F \cdot N \, dS = 4\pi.$

Ex 5 Prove that $\iint_S (\text{curl}\, F)\cdot N\, dS = 0$, assuming

F has continuous 2^{nd} partial derivatives and the surface S is smooth and bounds a finite region V.

Pf $\iint_S (\text{curl}\, F)\cdot N\, dS = \iiint_V \text{div}(\text{curl}\, F)\, dV$

$= \iiint_V 0\, dV = 0$ since $\nabla\cdot(\nabla\times F) = 0$.

Ex 6 Let F be a conservative divergence free vector field. Prove that

$$\iiint_V F\cdot F\, dV = \iint_{\partial V} f\, F\cdot N\, dS$$

where $F = \nabla f$ and ∂V, V have the usual nice properties.

Pf Divergence free means $\nabla\cdot F = 0$.

$\iint_{\partial V} (fF)\cdot N\, dS = \iiint_V \nabla\cdot(fF)\, dV$ by Div. Thm.

$= \iiint_V \underbrace{\nabla f}_{=F}\cdot F + f\, \cancel{\nabla\cdot F}^{\,0}\, dV$ by prod. rule

$= \iiint_V F\cdot F\, dV.$