Math 251

Practice Problems for Chapter 13

If you need help, send me your work and I will work with you. Some of these problems are quite challenging.

1. Let $\mathbf{F} = \left\langle x^3 + e^{-y}\sin z, x^2y + \arctan(z), \sqrt{y}\sec x\right\rangle$. Let V be the region of $\mathbb{R}^3$ bounded by the cylinder $z = 4 - x^2$, the plane $y + z = 5$, the xy-plane, and the xz-plane. Let S be the boundary of V. Find the flux of $\mathbf{F}$ out through S.
Answer: 4608/35.

2. Find the work done by the force field $\mathbf{F} = \langle y, z, x\rangle$ in moving a particle from $(0, 0, 0)$ to $(1, 1, 1)$ along the path $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$.
Answer: 89/60.

3. In this problem we study linear vector fields. These are vector fields of the form

$$\mathbf{F} = \langle ax + by + cz,\ dx + ey + fz,\ gx + hy + lz\rangle\,.$$

(a) Under what conditions on the coefficients will $\mathbf{F}$ be conservative? (I put this on a test some years ago, but nobody got it. See what you can do with it!)

(b) Suppose $\mathbf{F} = \langle 4x,\ -2y,\ 2z\rangle$. Find a potential function $f(x, y, z)$; *i.e.*, $\nabla f = \mathbf{F}$. What will type of surfaces will the level surfaces be?

4. Let $\mathbf{F} = \langle e^z, x^2, 3\rangle$. Let S be the portion of the ellipsoid $4x^2 + 4y^2 + 100z^2 = 100$ above the xy-plane. Find the flux of $\mathbf{F}$ up through S.
Hint: If $\mathbf{F}$ is divergence free, $\nabla \cdot \mathbf{F} = 0$, you can replace S with a simpler surface.
Answer: 75π

5. Find the surface area obtained by rotating the function $y = \sin x$ for $x \in [0, \pi]$ about the x-axis.
Hint: See Formula 10 in 13.6. The integration is very hard, but good for review.
The answer is ≈ 14.42.

6. Let $\mathbf{F} = \langle x^2z^3, 2xyz^3, xz^4\rangle$. Let S be the boundary of the unit cube, $[0, 1]\times[0, 1]\times[0, 1]$, in $\mathbb{R}^3$. Find the flux of $\mathbf{F}$ out through S.
Answer: 1.

7. Let R be a bounded region of $\mathbb{R}^3$ that satisfies the conditions for the divergence theorem. Let $f(x, y, z)$ be a scalar function with all its second derivatives continuous. Let $\mathbf{F} = \nabla f$ and assume $\nabla \cdot \mathbf{F} = 0$. Prove that

$$\iint_{\partial R} f\mathbf{F}\cdot\mathbf{N}\,dS = \iiint_R \mathbf{F}\cdot\mathbf{F}\,dV$$

where ∂R is the boundary of R.

8. Let $\mathbf{F}(x, y) = \langle xy^3 + x, x^2y^2 \rangle$. (a) Find the work done in pushing an object counterclockwise once around the square in the xy-plane with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$. (b) Find the work done in pushing an object counterclockwise once around the triangle in the xy-plane with vertices $(0, 0)$, $(1, 1)$, $(0, 1)$.
   Answers: (a) -1/6, (b) -1/10.

9. Compute $\oint_c (e^x \cos y)\, dx + (2x - e^x \sin y)\, dy$, where C is the boundary of the region in the xy-plane between the curves $y = x^2$ and $y = 8 - x^2$ going counterclockwise.
   Answer: 128/3.

10. Let S be a smooth surface enclosing a region R in $\mathbb{R}^3$. Let $f(x, y, z)$ be a scalar function with continuous second partial derivatives. Prove that

$$\iiint_R \nabla^2 f\, dV = \iint_S D_{\mathbf{N}} f\, dS,$$

   where $D_{\mathbf{N}} f$ is the directional derivative of f in the direction of the outward unit normal vector to S. Hint: $\nabla^2 f = \nabla \cdot (\nabla f)$ and $D_{\mathbf{N}} f = \nabla f \cdot \mathbf{N}$.

11. Let $\mathbf{F} = \langle 3z, 5x, -2y \rangle$. Let C be the ellipse formed from the intersection of of the plane $z = y + 3$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above. Compute $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$. Do this directly and by using Stokes' Theorem. *Answer:* 2π.