Frenet-Serret formulas and Torsion

We shall work through Problems 43, 45 and 46 in Section 10.8. These lead us to define the *torsion* of a space curve. At the end we discuss how torsion is a natural extension of the notions of *velocity* and *curvature*.

Recall the formulas in the box below.

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} \qquad \mathbf{B} \times \mathbf{T} = \mathbf{N} \qquad \mathbf{N} \times \mathbf{B} = \mathbf{T}$$

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w}$$

$$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$$

43. Show that $\dfrac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$.

Proof. By Definition $\boxed{8}$ $\left|\frac{d\mathbf{T}}{ds}\right| = \kappa$ so $\frac{d\mathbf{T}}{ds}$ will have the same magnitude as $\kappa\mathbf{N}$. Recall we defined $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$, but the parameterization used only affects the $\mathbf{N}$ by a scalar factor of ± 1. If arc length s is defined to be increasing with t then the sign choice is the same. Thus $\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{|\mathbf{T}'(s)|}$. Hence $\mathbf{N}(s)$ has the same direction as $\mathbf{T}'(s) = \frac{d\mathbf{T}}{ds}$ and by definition $\kappa \geq 0$. Thus, $\frac{d\mathbf{T}}{ds}$ and $\kappa\mathbf{N}$ have the same direction and magnitude, so they are equal. $\square$

45.a. Show that $\dfrac{d\mathbf{B}}{ds} \bullet B = 0$.

Proof. Since $|\mathbf{B}| = 1$ is a constant, Example 12 from 10.7 gives us that $B' \bullet B = 0$. $\square$

b. Show that $\dfrac{d\mathbf{B}}{ds} \bullet \mathbf{T} = 0$.

Proof.

$$\begin{aligned}
\mathbf{B}' \bullet \mathbf{T} &= (\mathbf{T} \times \mathbf{N})' \bullet T \\
&= (\mathbf{T}' \times \mathbf{N} + \mathbf{T} \times \mathbf{N}') \bullet \mathbf{T} \\
&= (\mathbf{T}' \times \mathbf{N}) \bullet \mathbf{T} + (\mathbf{T} \times \mathbf{N}') \bullet \mathbf{T} \\
&= (\kappa\mathbf{N} \times \mathbf{N}) \bullet \mathbf{T} - (\mathbf{N}' \times \mathbf{T}) \bullet \mathbf{T} \\
&= \kappa\mathbf{0} \bullet \mathbf{T} - \mathbf{N}' \bullet (\mathbf{T} \times \mathbf{T}) \\
&= 0 - \mathbf{N}' \bullet \mathbf{0} \\
&= 0
\end{aligned}$$

$\square$

c. Since $\frac{d\mathbf{B}}{ds}$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$, it must be parallel to $\mathbf{N}$. We define $-\tau(s)$ to be the scaling factor such that $\frac{d\mathbf{B}}{ds} = -\tau(s)\mathbf{N}$.

d. If a curve lives in a fixed plane then $\mathbf{B}$ is always the same unit vector perpendicular to that plane. Since $\mathbf{B}$ does not change $\frac{d\mathbf{B}}{ds} = 0$. Thus $\tau(s) = 0$ for such a curve. The reverse is true as well. More generally we can think of $\tau(s)$ to be a measure of the tendency of a curve to move away from its osculating plane.

46. Show that $\dfrac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}$.

Proof. From 43 we have $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$ and from 45c we know $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$.

$$\begin{aligned}
\mathbf{N}' = (\mathbf{B} \times \mathbf{T})' &= \mathbf{B}' \times \mathbf{T} + \mathbf{B} \times \mathbf{T}' \\
&= (-\tau\mathbf{N}) \times \mathbf{T} + \mathbf{B} \times (\kappa\mathbf{N}) \\
&= \tau(\mathbf{T} \times \mathbf{N}) - \kappa(\mathbf{N} \times \mathbf{B}) \\
&= \tau\mathbf{B} - \kappa\mathbf{T}
\end{aligned}$$

□

Summary. Let $\mathbf{r}(t)$ be a smooth space curve. The three equations in the box below are called the *Frenet-Serret formulas.*

$$\boxed{\begin{aligned}
\frac{d\mathbf{T}}{ds} &= \kappa\mathbf{N} \\
\frac{d\mathbf{N}}{ds} &= -\kappa\mathbf{T} + \tau\mathbf{B} \\
\frac{d\mathbf{B}}{ds} &= -\tau\mathbf{N}
\end{aligned}}$$

Sometimes they are expressed in matrix form.

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

- At any given time the curve has a location. The velocity v is the tendency to move away from the present location. If v is always 0 then we will stay at the our present location forever.
- At any given time the curve has a tangent line. The curvature κ is the tendency to move away from the present tangent line. If κ is always 0 then we will stay on the present tangent line forever.
- At any given time the curve has an osculating plane. The torsion τ is the tendency to move away from the present osculating plane. If τ is always 0 then we will stay on the present osculating plane forever.

However, velocity depends on the parameterization whereas curvature and torsion are geometric properties of the curve independent of the parameterization.