Vector Calculus Summary[1]

Types of Integrals. We can integrate along a curve C, over a surface S and through a 3-dimension region V. If the integrand is 1 we get arc length, surface area and volume respectively. If we are given some "density" function we can find the "mass".

Example 1. Let C be given by $r(t) = \langle t, t^2, t^3 \rangle$ for $0 \le t \le 1$. Let S be the portion of $z = f(x, y) = x^2 + 2y + 3$ over the unit disk U. Let V be the region inside the cylinder $x^2 + y^2 = 1$, below S and above the xy-plane. Find the length of C, the area of S and the volume of V.

Solution.

$$\text{Length} = \int_C ds = \int_0^1 |r'(t)|\, dt = \int_0^1 \sqrt{1 + 4t^2 + 9t^4}\, dt \approx 1.863022983 \quad \text{(done numerically)}.$$

$$\text{Surface area} = \iint_S dS = \iint_U \sqrt{1 + (z_x)^2 + (z_y)^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{1 + (2x)^2 + 2^2}\; r\, dr\, d\theta =$$

$$\int_0^{2\pi} \int_0^1 \sqrt{5 + 4r^2 \cos^2\theta}\, r\, dr\, d\theta \approx 7.670233535 \quad \text{(done numerically)}.$$

$$\text{Volume} = \iiint_V dV = \int_0^{2\pi} \int_0^1 \int_0^{r^2\cos^2\theta + 2r\sin\theta + 3} r\, dz\, dr\, d\theta =$$

$$\int_0^{2\pi} \int_0^1 (r^2\cos^2\theta + 2r\sin\theta + 3)\, r\, dr\, d\theta = \frac{13}{4}\pi.$$

Example 2. Use the same C, S and V. Let $h(x, y, z) = xyz$ be a density function. Find the masses of C, S and V. In the first case we assume h has units of mass per unit length, in the second mass per unit area and in the last mass per unit volume. Other applications might involve electric charge instead of mass.

Solution.

$$\text{Mass of C} = \int_C h\, ds = \int_0^1 h(t, t^2, t^3)|r'(t)|\, dt = \int_0^1 t^6 \sqrt{1 + 4t^2 + 9t^4}\, dt$$
$$\approx 0.442101217 \quad \text{(done numerically)}.$$

$$\text{Mass of S} = \iint_S h\, dS =$$

$$\iint_U h(x,y,x^2+2y+3)\sqrt{1+(z_x)^2+(z_y)^2}\,dA = 0 \text{ (done in the Appendix at the end).}$$

$$\text{Mass of V} = \iiint_V h(x,y,z)\,dV = \int_0^{2\pi}\int_0^1\int_0^{r^2\cos^2\theta+2r\sin\theta+3} h(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1\int_0^{r^2\cos^2\theta+2r\sin\theta+3}(r^2\cos\theta\sin\theta)z\,r\,dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1(r^2\cos\theta\sin\theta)(r^2\cos^2\theta+2r\sin\theta+3)\,r\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1 r^5\cos^3\theta\sin\theta+2r^4\cos\theta\sin^2\theta+3r^3dr\,d\theta =$$

$$\int_0^{2\pi}\frac{1}{6}\cos^3\theta\sin\theta+\frac{2}{5}\cos\theta\sin^2\theta+\frac{3}{4}\,d\theta = 0+0+\frac{3}{4}2\pi = \frac{3\pi}{2}$$

Integrals of Vector Fields: Work & Flux. If F is a vector field we can compute the work done as we travel along a curve C by taking the integral of the component of F tangent to C along C.

$$\text{Work} = \int_C F \bullet T\,ds.$$

We can compute the flux of F through S by using the surface integral over S of the component of F normal to S.

$$\text{Flux} = \iint_S F \bullet N\,dS.$$

Example 3. Using the same S and C as before and $F = \langle 3x+y, z, z^2\rangle$ find the work done along C and the flux through S.

Solution.

$$\text{Work} = \int_C F\bullet T\,ds = \int_0^1\left\langle 3t+t^2, t^3, t^6\right\rangle \bullet \frac{r'(t)}{|r'(t)|}|r'(t)|\,dt =$$

$$\int_0^1\left\langle 3t+t^2,t^3,t^6\right\rangle\bullet\left\langle 1,2t,3t^2\right\rangle\,dt = \int_0^1 3t+t^2+2t^4+3t^8\,dt =$$

$$3/2+1/3+2/5+3/7 = 559/210 \approx 2.6619047619.$$

To compute flux we need to figure out what to use for N and dS. To get N let $g(x,y,z) = z - f(x,y)$. Then let $N = \bigtriangledown g/|\bigtriangledown g| = \frac{-2x,-2,1}{\sqrt{4x^2+5}}$. As before $dS = \sqrt{f_x^2+f_y^2+1}dA = \sqrt{4x^2+5}dA$. Therefore,

$$\text{Flux} = \iint_S F \bullet N\, dS = \iint_U F \bullet \bigtriangledown g\, dA =$$

$$\iint_U \langle 3x + y, z, z^2 \rangle \bullet \langle -2x, -2, 1 \rangle\ dA = \iint_U -(3x+y)2x - 2z + z^2 dA =$$

$$\iint_U -(3x+y)2x - 2(x^2+2y+3) + (x^2+2y+3)^2\, dA =$$

$$\iint_U -2x^2 - 2xy + 8y + 3 + x^4 + 4x^2y + 4y^2\, dA =$$

$$\int_0^{2\pi} \int_0^1 (-2r^2\cos^2\theta - 2r^2\cos\theta\sin\theta + 8r\sin\theta + 3 + r^4\cos^4\theta + 4r^3\cos^2\theta\sin\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta =$$

$$\int_0^1 (-2\pi r^2 - 0 + 0 + 6\pi + r^4 3\pi/4 + 0 + 4\pi)\, r\, dr = \int_0^1 10\pi r - 2\pi r^3 + (3\pi/4) r^5\, dr =$$

$$5\pi - 2\pi/3 + 3\pi/5 = 74/15 \approx 4.9333$$

Integrating over Vector Fields: Short Cuts, FTC, Div Thm, Stokes' Thm.

If F is a **conservative** field, that is if there exists a scalar function f such that $F = \bigtriangledown f$, then the **Fundamental Theorem of Line Integrals** says

$$\int_C F \bullet T ds = \int_a^b \bigtriangledown f \bullet r'(t)\, dt = f(r(b)) - f(r(a))$$

where $r(t)$ with $a \le t \le b$ is a parametrization of C. This is called **path independence**. It follows that if C is a closed curve (loop) then

$$\oint_C F \bullet T ds = 0.$$

A vector field F is conservative if and only if $\bigtriangledown \times F = \mathbf{0}$.

(We drop the assumption that F is conservative.) If S is a surface with boundary C (a closed curve) then **Stokes' Theorem** says that

$$\oint_C F \bullet T\, ds = \iint_S (\bigtriangledown \times F) \bullet N\, dS.$$

Notice that if we change the surface S with another surface S' that has the same boundary C the result is unchanged. If S is a closed surface then C is a point and we get that the flux of the curl of F is zero though any closed surface.

In the plane Stokes's Theorem is becomes **Green's Theorem**.

If V is a connected region in with boundary S then the **Divergence Theorem** says if F has continuous derivatives in V then

$$\iint_S F \bullet N\, dS = \iiint_V \bigtriangledown \bullet F\, dV.$$

These three theorems are generalizations of the **Fundamental Theorem of Calculus**, which state that we can evaluate $\int_a^b f(x)\,dx$ from only knowing the anti-derivative on the boundary of the interval $[a,b]$; $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F'(x) = f(x)$. They can greatly simplify many calculations but also, with more experience, yield deep physical insights.

Example 4. Let V be the cube $[0,2]^3$ and let S be its boundary surface with outward normal. Let $F = \langle x^2, y^3, e^z - 1\rangle$. Find the flux of F through S.

Solution. By the divergence theorem

$$\iint_S F \bullet N dS = \iiint_V \triangledown \bullet F dV.$$

Now $\triangledown \bullet F = x^2 + y^3 + e^z - 1$. Thus

$$\text{flux} = \int_0^2 \int_0^2 \int_0^2 x^2 + y^3 + e^z - 1\,dx\,dy\,dz = 80 + 4(e^2 - 1).$$

To do this directly we'd need to integrate over each of the 6 faces separately. Yuck!

Example 5. Let $F = \langle 3x^2, 2y, 2z\rangle$. Find the work done in pushing a particle along the helix $\langle \cos \pi t, \sin \pi t, t^3\rangle$ from (1,0,0) to (1,0,8).

Solution I. Since $\triangledown \times F = \langle 0,0,0\rangle$ there is a potential function f with $\triangledown f = F$. In fact it is easy to see that $f(x,y,z) = x^3 + y^2 + z^2$ works. Thus,

$$\int_{\text{Helix}} F \bullet T\,ds = f(1,0,8) - f(1,0,0) = 64.$$

Solution II. If we wanted to do this directly we would have

$$\int_{\text{Helix}} F \bullet T\,ds = \int_0^2 \left\langle 3\cos^2 \pi t, 2\sin \pi t, 2t^3\right\rangle \bullet \left\langle -\pi \sin \pi t, \pi \cos \pi t, 3t^2\right\rangle\,dt =$$

$$\int_0^2 -3\pi \cos^2 \pi t \sin \pi t + 2\pi \cos \pi t \sin \pi t + 6t^5\,dt = 0 + 0 + 64.$$

Example 6. Let G be a vector field with all second partial derivatives continuous. Let S be a closed surface with outward normal. What is the flux of the curl of G out through S?

Solution. Let V be the bounded region in $\mathbb{R}^3$ with boundary S. Then

$$\iint_S \triangledown \times G\,dS = \iiint_V \triangledown \bullet (\triangledown \times G) dV = \iiint 0\,dV = 0.$$

Remember for any twice differentiable vector field G, div curl G = 0.

Example 7. Let $F = \langle x + y + 3z, 2x - z, x - 2y + z\rangle$. Let S be the surface of a bounded region V in $\mathbb{R}^3$ which has volume 23. Find the flux of F out through S.

Solution. Div F = 1+0+1 = 2. Thus

$$\text{flux} = \iint_S F \bullet N dS = \iiint_V 2dV = 2 \times 23 = 46.$$

Example 8. Let $F = \langle M, N, P \rangle = \left\langle x^3 y, xy^2, \sin \ln \sqrt{x + yz + 7} \right\rangle$. Let S be the portion of the cone $z = 9 - \sqrt{x^2 + y^2}$ above the xy-plane and below $z = 9$ with outward normal vector. Find the flux of $\bigtriangledown \times F$ through S.

Solution. Let C be the circle of radius 3 in the xy-plane with center at the origin, oriented counterclockwise. Then C is the boundary of S. By Stokes' Theorem

$$\iint_S \bigtriangledown \times F \bullet N dS = \oint_C F \bullet T ds.$$

Let S^* be the disk of radius 3 in the xy-plane with center at the origin with upward normal vector. Notice the boundary of S^* is just C. Thus by Stokes' Theorem (or Green's Theorem)

$$\oint_C F \bullet T ds = \iint_{S^*} \bigtriangledown \times F \bullet \langle 0, 0, 1 \rangle \, dA =$$

$$\iint_{S^*} M_y - N_x dA = \iint_{S^*} x^3 - y^2 dA =$$

$$\int_0^{2\pi} \int_0^3 (r^3 \cos^3 \theta - r^2 \sin^2 \theta) \, r \, dr \, d\theta = \int_0^3 0 - r^3 \pi \, dr = -81\pi/4.$$

Some Properties of Vector Fields

Let f be a scalar function. If all second partial derivatives are continuous then

$$\bigtriangledown \times (\bigtriangledown f) = \mathbf{0}$$

Let F be a vector field with continuous second partial derivatives. Then

$$\bigtriangledown \bullet (\bigtriangledown \times F) = 0.$$

As we stated above if $\bigtriangledown \times F = \mathbf{0}$, that is F is conservative, then there exists a scalar function f such that $F = \bigtriangledown f$. In many applications f is the potential energy (although physics books will let $U = -f$ and write $F = - \bigtriangledown U$). We developed a method for finding f.

If G is a vector field with $\bigtriangledown \bullet G = 0$, that is if G is divergence free, then there exists another vector field F such that $G = \bigtriangledown \times F$. In this case F is called a vector potential field for G. There is a method for finding F, but we have not covered this.

Example 9. A static electric force field is conservative and thus has a scalar potential function. This is not true for a magnetic field, but it turns out they are divergence free and so have vector potential field.

Appendix. Below is how I did the mass calculation for S in Example 2. This is pretty tricky and you would not have to do something like this on a test.

$$\text{Mass of S} = \iint_S h\,dS = \iint_U h(x,y,x^2+2y+3)\sqrt{1+(z_x)^2+(z_y)^2}\,dA =$$

$$\int_0^{2\pi}\int_0^1 h(r\cos\theta, r\sin\theta, r^2\cos^2\theta+2r\sin\theta+3)\sqrt{5+4r^2\cos^2\theta}\; r\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1 r\cos\theta r\sin\theta(r^2\cos^2\theta+2r\sin\theta+3)\sqrt{5+4r^2\cos^2\theta}\; r\,dr\,d\theta =$$

$$\int_0^1\int_0^{2\pi} r^4\cos^3\theta\sin\theta\sqrt{*}+2r^3\cos\theta\sin^2\theta\sqrt{*}+3r^2\cos\theta\sin\theta\sqrt{*}\;d\theta dr =$$

$$\int_0^1\int_0^{2\pi} r^2\cos\theta\sin\theta(r^2\cos^2\theta+3)\sqrt{*}+2r^3\cos\theta\sin^2\theta\sqrt{*}\;dr\,d\theta = 0.$$

To see this let

$$f(\theta) = \cos\theta\sin\theta E(\cos\theta) \quad \text{and} \quad g(\theta) = \cos\theta\sin^2\theta E(\cos\theta),$$

where E is any even integrable function. Check that $g(\theta+\pi) = -g(\theta)$. Then

$$\int_0^{2\pi} g(\theta)\,d\theta = \int_0^{\pi} g(\theta)\,d\theta + \int_{\pi}^{2\pi} g(\theta)\,d\theta = \int_0^{\pi} g(\theta)\,d\theta + \int_0^{\pi} g(\theta+\pi)\,d\theta = \int_0^{\pi} g(\theta)\,d\theta - \int_0^{\pi} g(\theta)\,d\theta = 0.$$

Next check that $f(\theta+\pi) = f(\theta)$ and $f(\pi-\theta) = -f(\theta)$. Now

$$\int_0^{2\pi} f(\theta)\,d\theta = 2\int_0^{\pi} f(\theta)\,d\theta = 2\left(\int_0^{\pi/2} f(\theta)\,d\theta + \int_{\pi/2}^{\pi} f(\theta)\,d\theta\right).$$

But, letting $\phi = \pi - \theta$ we have

$$\int_{\pi/2}^{\pi} f(\theta)\,d\theta = -\int_{\pi/2}^{0} f(\pi-\phi)\,d\phi = \int_0^{\pi/2} f(\pi-\phi)\,d\phi = -\int_0^{\pi/2} f(\phi)\,d\phi,$$

and the result follows.