3D Calculus Examples

1. Consider the solid paraboloid P given by $z = a^2 - x^2 - y^2$ and $z \geq 0$. Suppose the density is $\rho(x, y, z) = x^2 y^2 z$. Find the moment of inertia with respect to the z-axis.

   Answer: $\dfrac{a^{12}\pi}{960}$

2. Let C be the cube $[0, 2]^3$. Let $\mathbf{F} = \langle x^2, y^3, e^z - 1\rangle$. Find the flux of $\mathbf{F}$ out of C.

   Answer: $44 + 4e^2$

3. Let C be the circle of radius 4 with center $(0, 0)$ in the xy-plane. Let $\mathbf{F} = \langle x^2 - y, x - y^2\rangle$. Find the work done by $\mathbf{F}$ in pushing a particle around C once counter clockwise.

   Answer: 32π

4. Consider the closed loop L in $\mathbb{R}^3$ given by $\mathbf{r}(t) = \left\langle \cos^2 t, 5 + 3\sin 5t, \cos t \sin^2 t\right\rangle$ for $0 \leq t \leq 2\pi$. Let $\mathbf{F} = \left\langle x^2 \ln x, y^3 + 2\sin y, e^{z^2 - 2}\right\rangle$. Find the work by by $\mathbf{F}$ in pushing a particle around L once in the direction of increasing t.

   Solution: Show $\bigtriangledown \times \mathbf{F} = \mathbf{0}$. Thus no net work is done.

5. Consider the loop L formed by the intersection of the plane $x + z = 4$ and the cylinder $z = x^2 + y^2$. Let $\mathbf{F} = \langle y, 2x + y, x - y\rangle$. Find the work done by $\mathbf{F}$ in pushing a particle around C once, counterclockwise when viewed from above.

   Answer. 0

   Solution: $\bigtriangledown \times \mathbf{F} = \langle -1, -1, 1\rangle$. The unit normal vector to the plane pointing up is $\mathbf{N} = \langle 1, 0, 1\rangle / \sqrt{2}$. Thus $\bigtriangledown \times \mathbf{F} \cdot \mathbf{N} = 0$.

6. Let $\mathbf{F} = \langle 4x + y, y^2 + 3x\rangle$. Let C be the cardioid $r = 1 + \cos\theta$, $0 \leq \theta \leq 2\pi$. Find the work done by $\mathbf{F}$ in moving a particle once around C counterclockwise.

   Answer. 3π

7. Consider the solid paraboloid P given by $z = a^2 - x^2 - y^2$ and $z \geq 0$. Derive formulas for the volume and surface areas.

   Answers. $\frac{\pi}{2}a^4$ & $\frac{\pi}{6}\left(\left(\sqrt{4a^2 + 1}\right)^3 - 1\right)$

8. Let $\mathbf{F} = \langle 0, 0, z^2\rangle$. Let S be the upper hemisphere of the unit sphere centered at the origin of $\mathbb{R}^3$. Find the flux of $\mathbf{F}$ up through S.

   Answer. $\frac{\pi}{2}$

9. Let C be the portion of the helix

$$\mathbf{r}(t) = \langle \cos 2\pi t, \sin 2\pi t, t\rangle$$

   for $0 \leq t \leq 1$. Let

$$\mathbf{F} = \langle z + 4yz + y\cos(xy), 4xz + x\cos(xy), x + 4xy\rangle .$$

Find the work done by $\mathbf{F}$ in pushing a particle along C in the direction of increasing t.

Answer. 1

10. Let $\mathbf{F} = \langle x+y, y+z, x+z \rangle$. Let R be the cylindrical solid bounded by $z = 0$, $z = 5$, and $x^2 + y^2 = 9$. Find the flux of $\mathbf{F}$ out through the boundary of R.

    Answer. 135π

11. Consider a solid cylinder of height h and radius a with density proportional to the distance from the base.
    a. Find the total mass.
    b. Find the center of mass.
    c. Find moment of rotational inertia with respect to its axis.

    Answers. $\frac{1}{2}ka^2h^2\pi$, $(0, 0, 2h/3)$, $\frac{1}{4}a^4h^2\pi$.

12. Let S be the closed surface formed from the portion of the paraboloid $z = 9 - x^2 - y^2$ above the xy-plane and the disk of radius 3 center (0,0) in the xy-plane. Let $\mathbf{F} = \langle 7xy, z, 4xy + y^2 \rangle$. Find the flux of $\mathbf{F}$ out through S.

    Answer. 0

13. Let C be the boundary of the rectangle in the $z = y$ plane determined by $0 \leq x \leq 1$ and $0 \leq y \leq 3$. Let $\mathbf{F} = \langle x^2, 4xy^3, xy^2 \rangle$. Find the work done by $\mathbf{F}$ in pushing a particle around C counterclockwise when viewed from above.

    Answer. 90

14. Let C be the triangle in $\mathbb{R}^3$ with vertices $(a, 0, 0)$, $(0, a, 0)$ and $(0, 0, a)$ oriented clockwise when viewed from above. Evaluate the line integral

    $$\int_C y^2\,dx + z^2\,dy + x^2\,dz.$$

    Answer. a^3

15. Let $\mathbf{F} = \langle 2xyz, x^2z, x^2y \rangle$. Show that $\mathbf{F}$ is conservative and find a potential function for $\mathbf{F}$. If C is any path from $(0, 0, 0)$ to $(1, 2, 3)$ evaluate $\int_C \mathbf{F} \cdot \mathbf{T}ds$.

    Answer. 6