Math 251 Test 1 Spring 2012

Name: Key

SCIENTIFIC CALCULATORS ONLY

1. [15 points] The Law of Cosines states that for a triangle with side lengths A, B and C with θ the angle between sides A and B we have

$$C^2 = A^2 + B^2 - 2AB\cos\theta.$$

Use this fact to derive the following identity for vectors.

$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$

where $\mathbf{u}$ and $\mathbf{v}$ vectors with a common base point and θ is the angle between them.

See page 572 in textbook.
(Fig 1 is on the previous page.)

2. [20 points] Let $\mathbf{u} = \langle 1, 1, 1\rangle$, $\mathbf{v} = \langle 0, 2, 2\rangle$, and $\mathbf{w} = \langle -1, 0, 3\rangle$.

(a) Find $\mathbf{u} \times (\mathbf{w} + \mathbf{v})$.

$$w + v = \langle -1, 2, 5\rangle$$

$$u \times (w+v) = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -1 & 2 & 5 \end{vmatrix} = \langle 3, -6, 3\rangle$$

(b) Find the angle between $\mathbf{u}$ and $\mathbf{v}$ in degrees.

$$\cos\theta = \frac{u \cdot v}{|u||v|} \qquad u \cdot v = \langle 1,1,1\rangle \cdot \langle 0,2,2\rangle = 2+2 = 4$$

$$|u| = \sqrt{1^2+1^2+1^2} = \sqrt{3} \qquad |v| = \sqrt{0^2+2^2+2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\cos\theta = \frac{4}{\sqrt{3}\cdot 2\sqrt{2}} = \sqrt{\frac{2}{3}} \qquad \theta = \cos^{-1}\left(\sqrt{\tfrac{2}{3}}\right) \approx \underline{35.26^\circ}$$

(c) Find the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

$$\text{Area} = |u \times v| = \sqrt{0^2+(-2)^2+2^2} = \sqrt{8} = \underline{2\sqrt{2}}$$

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix} = \langle 0, -2, 2\rangle$$

(d) Cross out the expressions below which are not defined.

$\mathbf{v} + 7\mathbf{u}$ $\quad 8 + \mathbf{w}$ $\quad 7\mathbf{v} + \mathbf{u} \bullet \mathbf{u}$ $\quad \mathbf{v} \times \mathbf{u} \times \mathbf{w}$ $\quad (\mathbf{v} \times \mathbf{u}) \times \mathbf{w}$

number plus a vector

ambiguous since cross prod. is not assoc.

3. [15 points] Let $\mathbf{v}(t) = \langle 1, 3\sqrt{t}, 3\sqrt{t}\rangle$ and $\mathbf{r}_0 = \langle 1,2,3\rangle$ be the velocity and starting position of a particle traveling in $\mathbb{R}^3$.

(a) Find position vector function $\mathbf{r}(t)$.

$$r(t) = \int v(t)\,dt = \left\langle t + C_1,\ 3\left(\tfrac{2}{3}t^{3/2}\right) + C_2,\ 3\left(\tfrac{2}{3}t^{\frac{3}{2}}\right) + C_3 \right\rangle$$
$$= \langle t+1,\ 2t^{3/2}+2,\ 2t^{3/2}+3\rangle$$

(b) Find the arc length the particle will travel from $t = 0$ to $t = 1$.

$$L = \int_0^1 |v(t)|\,dt = \int_0^1 \sqrt{1^2 + 9t + 9t}\;dt = \int_0^1 \sqrt{1+18t}\,dt$$

Let $u = 1+18t$.
Then $du = 18\,dt$
or $dt = \frac{du}{18}$

$$= \int_1^{19} u^{\frac{1}{2}} \frac{du}{18} = \frac{1}{18}\left(\frac{u^{3/2}}{\frac{3}{2}}\right) = \frac{1}{27}\left(19^{3/2} - 1\right)$$
$$\approx 3.03039$$

(c) Recall the curvature is $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$. Find $\kappa(1)$.

$$r''(t) = v'(t) = \left\langle 0,\ \tfrac{3}{2}t^{-\frac{1}{2}},\ \tfrac{3}{2}t^{-\frac{1}{2}}\right\rangle \qquad r''(1) = \left\langle 0, \tfrac{3}{2}, \tfrac{3}{2}\right\rangle$$
$$= \tfrac{3}{2}\langle 0,1,1\rangle$$
$$r'(t) = v(t) = \left\langle 1,\ 3t^{\frac{1}{2}},\ 3t^{\frac{1}{2}}\right\rangle \qquad r'(1) = \langle 1,3,3\rangle$$

$$|r'(1)| = \sqrt{1+9+9} = \sqrt{19}$$

$$r'(1)\times r''(1) = \frac{3}{2}\begin{vmatrix} i & j & k \\ 1 & 3 & 3 \\ 0 & 1 & 1 \end{vmatrix} = \frac{3}{2}\langle 0,-1,1\rangle$$

$$|r'(1)\times r''(1)| = \frac{3}{2}\sqrt{0^2+(-1)^2+1^2} = \frac{3\sqrt{2}}{2}$$

$$\kappa(1) = \frac{\frac{3\sqrt{2}}{2}}{19\sqrt{19}} \approx 0.256$$

4. [4+4+2 points] Let $f(x,y,z) = \dfrac{x^2+2yz}{3x^2+y^2+z^2}$.

(a) Find the limit of $f(x,y,z)$ as we approach the origin along the positive x-axis.

$y=0, z=0$ limit as $x \to 0^+$.

$$\lim_{x\to 0^+} f(x,0,0) = \lim_{x\to 0^+} \frac{x^2+0}{3x^2+0+0} = \lim_{x\to 0^+} \frac{x^2}{3x^2} = \lim_{x\to 0^+} \frac{1}{3} = \frac{1}{3}$$

(b) Find the limit of $f(x,y,z)$ as we approach the origin along the positive y-axis.

$x=0, z=0$ $y\to 0^+$

$$\lim_{y\to 0^+} f(0,y,0) = \lim \frac{0+0}{0+y^2+0} = \lim_{y\to 0^+} 0 = 0$$

(c) What can you conclude about the $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z)$?

Does not exist.

5. [10 points] The acceleration vector of a particle moving in $\mathbb{R}^3$ is given by

$$\mathbf{a}(t) = \langle 2,-2,1\rangle .$$

Suppose that the particle starts from the origin at rest. How long will it take for the particle to travel 6 units?

$$v(t) = \int_0^t a(t)\,dt = \langle 2t+c_1, -2t+c_2, t+c_3\rangle = \langle 2t,-2t,t\rangle.$$

$$\text{dist. traveled} = \int_0^t |v(t)|\,dt = \int_0^t \sqrt{4t^2+4t^2+t^2}\,dt = \int_0^t 3t\,dt$$

$$= \frac{3}{2}t^2\Big|_0^t = \frac{3}{2}t^2.$$

Let $\frac{3}{2}t^2 = 6.$

$t^2 = 4$

$t = 2$

6. [10+2+3 points] Let $f(x, y) = ye^x$.

(a) Plot the level curves for $f(x, y) = 0, \pm 1,$ and ± 2 on the grid below.

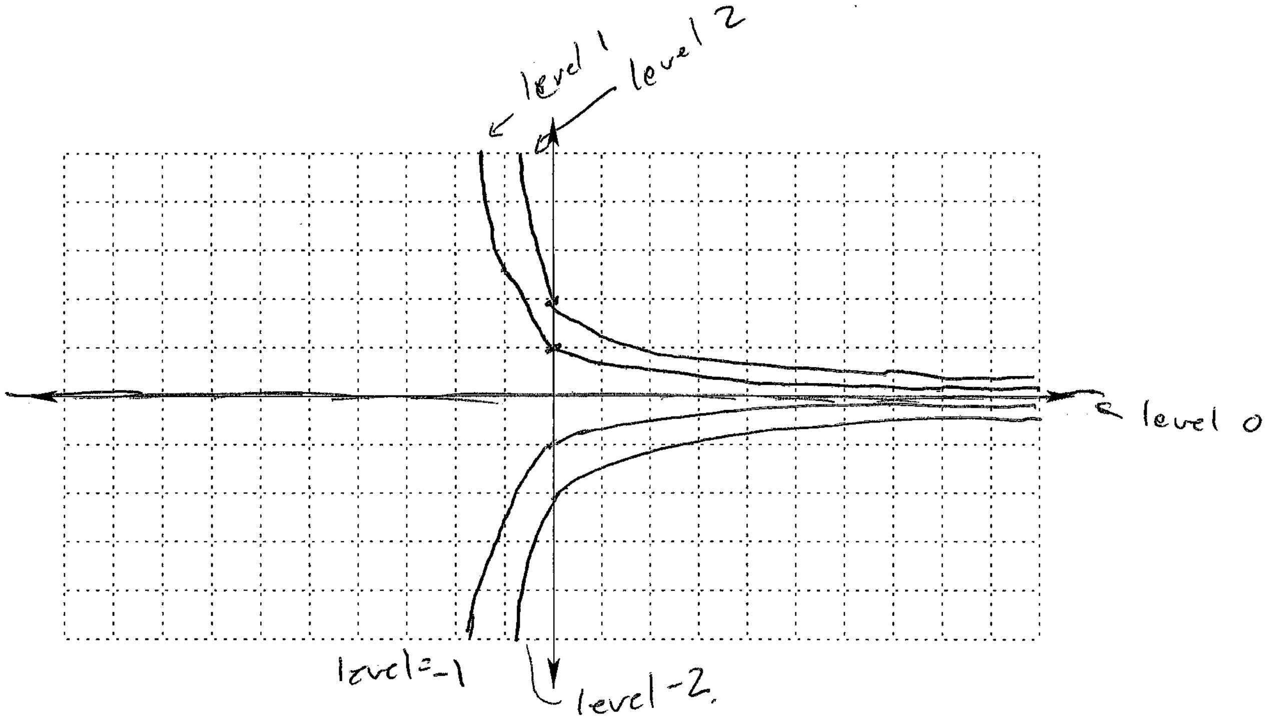
$ye^x = 0 \Rightarrow y = 0$ (x-axis)

$ye^x = 1 \Rightarrow y = e^{-x}$

$ye^x = 2 \Rightarrow y = 2e^{-x}$

$ye^x = -1 \Rightarrow y = -e^{-x}$

$ye^x = -2 \Rightarrow y = -2e^{-x}$



(b) Find $\frac{\partial f}{\partial x}$

ye^x

(c) Find f_{xy}.

e^x

7. [15 points] At what point do the curves

$$\mathbf{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle$$

and

$$\mathbf{r}_2(s) = \langle 3-s, s-2, s^2 \rangle$$

intersect? If these equations represent the motion of two objects, do they collide?

$t = 3-s$
$1-t = s-2$) same.
$3+t^2 = s^2$

plug $t = 3-s$ into last eq

$3 + (3-s)^2 = s^2$

$3 + 9 - 6s + \cancel{s^2} = \cancel{s^2}$

$12 - 6s = 0$

$s = 2.$

when $s = 2$, $t = 3-2 = 1.$

So they do not

$r_1(1) = \langle 1, 0, 4 \rangle$

$r_2(2) = \langle 1, 0, 4 \rangle$

This is where the two curve intersect.

8. [10 BONUS points] Let $\mathbf{r}(t) = \langle 5\sin(e^t), 7, 5\cos(e^t) \rangle$. Find (a) the curvature and (b) the osculating plane for any time t. Hint: There is an easy way to do find these and a hard way. The hard will take forever and there is no partial credit.

$r(t)$ traces out a circle of radius 5 in the plane $y=7$.

Thus the curvature is $\frac{1}{5}$ and the osc. plane is $y=7$.