Math 251 Test 2 Spring 2012

Name: Key

SCIENTIFIC CALCULATORS ONLY

1. [10+5 points] Let $f(x,y,z) = x^2ye^z$. (a) Find the direction and magnitude of the **maximum** rate of change of f at the point $(2,1,0)$. (b) What is the rate of change of f in the direction of $\mathbf{v} = \langle 0,3,1\rangle$ at the point $(2,1,0)$.

$$\nabla f = \langle 2xye^z, x^2e^z, x^2ye^z\rangle = \langle 4, 4, 4\rangle.$$

$$|\nabla f| = \sqrt{4^2+4^2+4^2} = \sqrt{48} = 4\sqrt{3}$$

So the direction of the max rate of change is $\langle 4,4,4\rangle$ and its mag. is $4\sqrt{3}$.

$\nabla f \cdot \frac{\langle 0,3,1\rangle}{|\langle 0,3,1\rangle|} = \frac{0+12+4}{\sqrt{9+1}} = \frac{16}{\sqrt{10}}$ is the rate of change in the direction of $\mathbf{v}$.

2. [10 points] Let $f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$. Find and classify the critical points as relative maximums, minimums or saddle points.

Find cr. pts.

$f_x = 4x + 2y + 2 = 0$

$f_y = 2x + 2y = 0 \Rightarrow x = -y$

$-2y + 2 = 0$

$y = 1, x = -1$ is only cr. pt.

Use 2nd Der Test

$f_{xx} = 4 > 0$

$f_{yy} = 2$

$f_{xy} = 2$

$D = 4\cdot 2 - 2^2 = 4 > 0$

Thus $(-1,1)$ is a local min. $f(-1,1) = -4$.

3. [15 points] Maximize the function $f(x,y) = \sqrt{6-x^2-y^2}$ subject to the constraint $x+y=2$.

I'll use L. multipliers although older methods work too.

Let $g = x+y$.

$\nabla f = \left\langle \frac{-2x}{2\sqrt{\ }}, \frac{-y}{\sqrt{\ }} \right\rangle \qquad \nabla g = \langle 1, 1 \rangle.$

$\frac{-x}{\sqrt{\ }} = \lambda \cdot 1$

$\frac{-y}{\sqrt{\ }} = \lambda \cdot 1$

$\Rightarrow \frac{-y}{\sqrt{\ }} = \frac{-y}{\sqrt{\ }} \Rightarrow x = y.$

$x+y=2$

$x=1, y=1.$

$f(1,1) = 2$

4. [15 points] Find an equation for the tangent plane to the surface $xyz + z^2y - z = 1$ at (1,1,1).

Let $g = xyz + z^2y - z$.

$\nabla g = \langle yz, xz + z^2, xy + 2zy - 1 \rangle$

$= \langle 1, 2, 3 \rangle.$

This will be normal to the tang. plane.

$(x-1) + (y-1) + (z-1) = 0$

$\boxed{x+y+z=3}$

5. [15 points] Find the volume under the surface $z = xy + 5$ and above the disk in the xy-plane of radius 2 centered at the origin.

$$V = \iint_{disk} xy+5 \, dA = \int_0^{2\pi}\int_0^2 (\underbrace{r\cos\theta \, r\sin\theta}_{0} + \underbrace{5}_{0}) \, r\,dr\,d\theta$$

$$= 2\pi \cdot 5 \int_0^2 r\,dr = 10\pi \cdot \frac{2^2}{2} = 20\pi$$

Notice $20\pi = 5\cdot(4\pi) = 5\cdot$ area of the disk
and 5 is the average height!

6. [10 points] A function $u(x, y)$ is a solution to Laplace's equation if $u_{xx} + u_{yy} = 0$. Let

$$u(x, y) = e^x \sin y.$$

Is this function a solution to Laplace's equation? Show your work.

$$u_{xx} = (e^x \sin y)_x = e^x \sin y$$

$$u_{yy} = (e^x \cos y)_y = -e^x \sin y.$$

Thus $u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0.$

✓

7. [20 points] A disk of uniform density with mass 10 kg and radius 1 meter is to spin at 5 revolutions per second about its center. How much energy will this require, ignoring friction? Recall: K.E. $= \frac{1}{2}I\omega^2$. So you need to find I, the inertia about the center.

Density is $\frac{10}{\pi r^2} = \frac{10}{\pi}$

$$I = \iint_{\text{disk}} r^2 \frac{10}{\pi}\, dA = \frac{10}{\pi}\int_0^{2\pi}\int_0^1 r^2 \cdot r\, dr\, d\theta$$

$$\frac{10}{3\pi}\int_0^{2\pi} d\theta = \frac{20\pi}{3\pi} = \frac{20}{3}.$$

$$K.E. = \frac{1}{2}\cdot\frac{20}{3}\cdot(5\cdot 2\pi)^2 = \frac{10}{3}\cdot 100\pi^2$$

$$= \frac{1000}{3}\pi^2$$