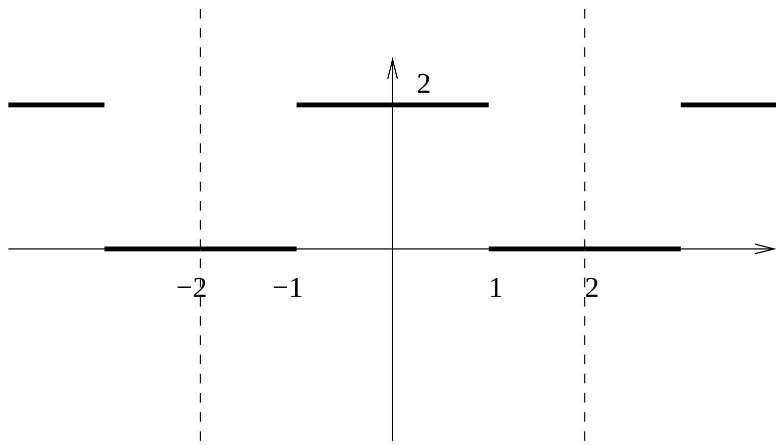
Final Exam **Math 305** **Spring Semester, 1997**

1. [25 points] Let $f(x)$ be a periodic function defined by the graph below. Find a_0, a_1, and a_2.



2. [25 points] Find the solution of the heat conduction problem

$$100u_{xx} = u_t, \qquad 0 < x < 1, \qquad t > 0$$
$$u(0,t) = 0, \qquad u(1,t) = 0 \qquad t > 0$$
$$u(x,0) = \sin(2\pi x) - 2\sin(5\pi x), \qquad \leq x \leq 1.$$

3. [25 points] Let $2y'' + y' + xy = 0$. Let $y = \sum_{n=0}^{\infty} a_n y^n$ be the solution. Assuming a_0 and a_1 are given, find a_2 and a_3 in terms of a_0 and a_1.

4. [25 points]

   a. Draw the direction field of $y_1' = \dfrac{3 - y_1}{2}$. Draw some solution curves.

   b. Draw the direction field of $y_2' = \left(\dfrac{3 - y_2}{2}\right) x$. Draw some solution curves.

   c. Find $\lim_{x\to\infty} y_1(x)$ and $\lim_{x\to\infty} y_2(x)$. Which converges faster? EXPLAIN.

   d. In each case, what happens as $x \to -\infty$?

5. [25 points] Solve each of the following differential equations.

   a. $\dfrac{dy}{dx} = -\dfrac{2xy + y^2}{x^2 + 2xy}$. DO NOT SOLVE FOR y. Hint: check for exactness.

   b. $(e^x + 1)\dfrac{dy}{dx} = y - ye^x$. Solve for y. Hint: Multiply both sides by $e^{-x/2}$. The integration will be easier.

   c. $xy' = y + xe^{(y/x)}$. Assume $x > 0$. Solve for y. Hint: Let $v = y/x$.

   d. $\dfrac{dy}{dx} + y = \dfrac{1}{1 + e^x}$. Solve for y.

6. [25 points] A body of mass m falls from rest in a medium offering resistance proportional to the square of the velocity. Find the relation between the velocity v and the time t. Find the limiting velocity, v_l.

   Hint:
$$\int \frac{dx}{a^2 - b^2x^2} = \frac{1}{2ab} \ln \left| \frac{a + bx}{a - bx} \right| + C.$$

7. [25 points] Find the general solution to $y'' - 2y' = \sin x$.

8. [25 points] The motion of a certain spring-mass systems is governed by the differential equation
$$u'' + 0.125u' + u = 0,$$
   where u is in feet and t in seconds. If $u(0) = 2$ and $u'(0) = 0$, determine the position of the mass at any time. (I.e. solve for $u(t)$.)

9. **BONUS PROBLEM [25 points].** The number of algae cells in a tank of water grows according to
$$\begin{aligned} A'(t) &= .2\left(1 - \frac{A(t)}{100}\right)A(t) && \text{light on, and} \\ A'(t) &= -.2A(t) && \text{light off.} \end{aligned}$$
   In words, the carrying capacity drops from 100 (billion cells) to zero when the light goes out. At $t = 0$ you start a ten (10) hour experimental run with $A(0) = 25$ and plan to keep the light on. When you come back at $t = 10$ you discover that the light bulb has burned out. You measure $A(10)$ to be 15. What time did the light bulb burn out?

   Hints: You can use your graphing calculator, but make sure your answer in correct to at least 4 decimal places. The integral below will be helpful:
$$\int \frac{dx}{x(a + bx)} = \frac{1}{a} \ln \left| \frac{x}{a + bx} \right| + C$$