

The Heat Equation Summary of Solutions

The One-dimensional Heat equation is

$$\alpha^2 U_{xx} = U_t,$$

where $U(x, t)$ is a function of length x and time t . We assume the initial temperature distribution is given by a function $f(x)$. That is,

$$U(x, 0) = f(x)$$

We have considered the Heat Equation with three different boundary (end point) conditions.

1. $U(0, t) = U(L, t) = 0$ for $t > 0$.
2. $U(0, t) = T_1$ and $U(L, t) = T_2$ for $t > 0$.
3. $U_x(0, t) = U_x(L, t) = 0$ for $t > 0$

We arrived at the following respective solutions.

1. $U(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2 \alpha^2 \pi^2 t}{L^2}}$, where $b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$.
2. Let $v(x) = \frac{T_2 - T_1}{L}x + T_1$. Then $U(x, t) = v(x) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2 \alpha^2 \pi^2 t}{L^2}}$,
where $b_n = \frac{2}{L} \int_0^L (f(x) - v(x)) \sin\left(\frac{n\pi x}{L}\right) dx$.
3. $U(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2 \alpha^2 \pi^2 t}{L^2}}$, where $a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$.

Ideally you should understand the derivations of these results. But, even if some of the details went by you, you should take note of the following. In each case the limit as $t \rightarrow \infty$ behaves as your intuition says it should. You should check that in each case $U(x, 0)$ is the Fourier Series of the Odd Periodic extension of $f(x)$. Also notice that each term of the summands satisfies the Heat Equation. So, by linearity and some facts about convergence (that we did not cover) The infinite series are solutions.