Homework Set 12. Due Friday December 3.

The wave equation with damping is $a^2 u_{xx} = u_{tt} + \gamma u_t$. The boundary conditions are still $u(0,t) = u(L,t) = 0$ for all times t. The initial conditions are $u(x,0) = f(x)$ for some given function $f(x)$, and $u_t(x,0) = 0$.

We will use $a = \gamma = 1$ and $L = 2$ with

$$f(x) = \begin{cases} x & \text{for} \quad x \in [0,1) \\ 2 - x & \text{for} \quad x \in [1,2] \end{cases}$$

Recall that in class we found the Fourier Sine Series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} \frac{8\sin(\frac{n\pi}{2})}{n^2\pi^2} \sin(\frac{n\pi x}{2}).$$

1. Verify that $a^2 u_{xx} = u_{tt} + \gamma u_t$ is linear. That is, show that if u_1 and u_2 are solutions, then so is $u = C_1 u_1 + C_2 u_2$.

2. Suppose $u(x,t) = X(x)T(t)$, and show this leads to

$$\begin{array}{ccc} X'' + \sigma X = 0 & & T'' + T' + \sigma T = 0 \\ & \text{and} & \\ X(0) = X(2) = 0 & & T'(0) = 0 \end{array}$$

3. Clearly $X(x)$ is just as in class and must have $\sigma = \frac{n^2\pi^2}{4}$, for $n = 1, 2, 3, \ldots$ in order to satisfy the boundary conditions. Show that for each n,

$$T_n(t) = e^{-t/2}\left(\sin\left(t\frac{\sqrt{n^2\pi^2 - 1}}{2}\right) + \sqrt{n^2\pi^2 - 1}\cos\left(t\frac{\sqrt{n^2\pi^2 - 1}}{2}\right)\right),$$

   solves the T initial value problem.

4. By linearity (and some facts about limits) $u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t)$ will be a solution to the PDE that satisfies the boundary conditions, provided it converges. Check that $u_t(x,0) = 0$.

5. Use $u(x,0) = f(x) =$ the Fouries Series of the odd periodic extention of $f(x)$ that we found in class to show that,

$$c_n = \frac{8\sin(\frac{n\pi}{2})}{n^2\pi^2\sqrt{n^2\pi^2 - 1}}$$

   Rewright this expression without using trig functions.

6. [Optional] Use Maple to animate your result. Either e-mail your Maple worksheet to me, or come by and show me.