Math 305 Test 3 Fall 2012

Name: Solutions Section time: ______

SCIENTIFIC CALCULATORS ALLOWED

1. [20 points] Suppose $y'' + (\tan t)y' + ty = 0$ and that $y(0) = 2$ and $y'(0) = 0$.

   a. Find the first five terms of the power series solution for $y(t)$ centered about $t = 0$.

$$a_n = \frac{y^{(n)}(0)}{n!}$$

$a_0 = 2$, $a_1 = 0$.

a_2 : $y''(\cancel{t}) = -\tan t\, y' - t y$

$y''(0) = 0 - 0 = 0 \Rightarrow a_2 = 0$

a_3: $y''' = -\sec^2 t\, y' - \tan t\, y'' - y - t y'$

$y'''(0) = -1 \cdot 0 - 0 \cdot 0 - 2 - 0 \cdot 0 = -2$

$a_3 = \frac{-2}{3!} = \frac{-1}{3}$

a_4 : $y'''' = -2\sec^2 t \tan t\, y' - \sec^2 t\, y'' - \sec^2 y''$
$- \tan t\, y''' - y' - y' - t y''$

$y''''(0) = 0 - 1 \cdot 0 - 1 \cdot 0 - 0 - 0 - 0 = 0$

$a_4 = 0$

   b. Give a lower bound for the radius of convergence.

$y(t) \approx 2 - \frac{1}{3}X^3$

$\tan t$ is singular at $\pm \frac{\pi}{2}$.

dist from 0 is $\frac{\pi}{2}$.

so rad. $\geq \pi/2$.

2. [20 points] Consider $y'' + xy' - 2y = 0$.

a. Find the general power series solution including a recursive formula for a_n.

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \qquad xy' = \sum_{n=0}^{\infty} n a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n - 2 a_n \right] x^n = 0$$

$$a_{n+2} = \frac{(2-n) a_n}{(n+2)(n+1)} \qquad n \geq 0$$

$$a_n = \frac{(4-n) a_{n-2}}{n(n-1)} \qquad n \geq 2$$

b. Now suppose $y(0) = 1$ and $y'(0) = 0$. Find the values of all the a_n's. (The series should terminate quickly.)

$$a_0 = 1 \qquad a_1 = 0. \qquad a_2 = \frac{+2 a_0}{2 \cdot 1} = a_0 = 1$$

$$a_3 = \frac{3 \cdot a_1}{3 \cdot 2} = 0 \qquad a_4 = \frac{(4-4) a_2}{4 \cdot 3} = 0$$

$$a_5 = \frac{1 \cdot a_3}{5 \cdot 4} = 0 \qquad a_6 = \frac{-2 \cdot a_4}{6 \cdot 5} = 0$$

$$a_7 = \frac{3 \cdot a_5}{7 \cdot 6} = 0 \qquad a_8 = \frac{-4 \cdot a_6}{8 \cdot 7} = 0$$

⋮

all odd terms are zero

all remaining even terms are zero.

$$y = 1 + x^2$$

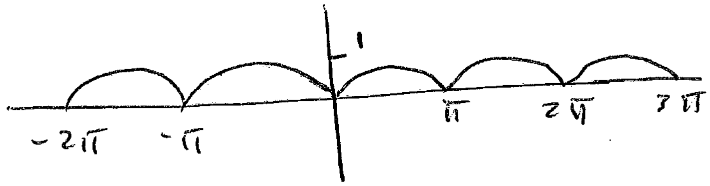
(plug in and check it!)

3. [20 points] Consider the wave $f(x) = |\sin x|$.

a. Graph this function for several periods. What is the period? Is it even, odd or neither? Even

the function

Period is π.
We can use $L = \pi/2$ or $L = \pi$.



b. Set up the integrals needed to find the Fourier Series of $f(x)$.

Since $f(x)$ is even all $b_n = 0$.

$L = \pi$ $\quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin x|\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi}$

$L = \pi/2$ $\quad a_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} |\sin x|\,dx = \frac{4}{\pi}\int_0^{\pi/2} \sin x\,dx = \frac{4}{\pi}$

For the other a_n's the results are different depending on whether $L = \pi/2$ or π.

c. Find a_0, a_2 and b_{13}. (The a_n's are the cosine coefficients and the b_n's are the sine coefficients as in the text and in class.) Hint: $\cos 2x = \cos^2 x - \sin^2 x$.

I'll do $L = \pi$ here and $L = \pi/2$ on the next page.

In either case for $n \geq 1$ $\quad a_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)|\sin x|\,dx$

$L = \pi$

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\left(\frac{2\pi x}{\pi}\right)|\sin x|\,dx = \frac{2}{\pi}\int_0^{\pi} \cos(2x)\sin x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} c^2 s - s^3\,dx = \frac{2}{\pi}\int_0^{\pi} c^2 s - (1-c^2)s\,dx = \frac{2}{\pi}\int_0^{\pi} (2c^2-1)s\,dx$$

Let $u = \cos x$. $du = -\sin x$. Thus $a_n = \frac{-2}{\pi}\int_1^{-1} 2u^2-1\,du = \frac{4}{\pi}\int_0^1 2u^2-1\,du$

$$= \frac{4}{\pi}\left(\frac{2u^3}{3} - u\right)\Big|_0^1 = \frac{4}{\pi}\left(\frac{2}{3} - 1\right) = \frac{-4}{3\pi}.$$

#3 cont. $L = \pi/2$.

$$a_2 = \frac{1}{\pi/2}\int_{-\pi/2}^{\pi/2} \cos\left(\frac{2\pi x}{\pi/2}\right)|\sin x|\,dx = \frac{4}{\pi}\int_0^{\pi/2} \cos(4x)\sin(x)\,dx$$

$$\cos(4x) = \cos^2(2x) - \sin^2(2x) = [c^2 - s^2]^2 - 4s^2c^2 \qquad s^2 = 1 - c^2$$

$$= [2c^2-1]^2 - 4(c^2 - c^4) = 4c^4 - 4c^2 + 1 - 4c^2 + 4c^4$$

$$= 8c^4 - 8c^2 + 1$$

$$a_2 = \frac{4}{\pi}\int_0^{\pi/2} (8c^4 - 8c^2 + 1)\, s\, dx = -\frac{4}{\pi}\int_1^0 8c^4 - 8c^2 + 1\, dc$$

$$dc = -s\,dx$$

$$= \frac{4}{\pi}\left(\frac{8c^5}{5} - \frac{8c^3}{3} + c\right)\Bigg|_0^1 = \frac{4}{\pi}\left(\frac{8}{5} - \frac{8}{3} + 1\right) = \frac{4}{\pi}\left(\overset{-16}{\frac{24-40}{15}} + \frac{15}{15}\right) = \frac{-4}{15\pi}.$$

For $L = \pi$

$a_1 = 0$

$a_2 = \frac{-4}{3\pi}$

$a_3 = 0$

$a_4 = \frac{-4}{15\pi}$

$a_5 = 0$

$\vdots$

$$a_n = \frac{-2(1+\cos(n\pi))}{(n^2-1)\pi} \begin{cases} 0 & \text{odd} \\ \dfrac{-4}{(n^2-1)\pi} & \text{even} \end{cases}$$

For $L = \frac{\pi}{2}$

$a_1 = \frac{-4}{3\pi}$

$a_2 = \frac{-4}{15\pi}$

$a_3 = \frac{-4}{35\pi}$

$\vdots$

$$a_n = \frac{-4}{(\underbrace{4n^2}_{(2n)^2}-1)\pi}$$

So, the two options for L do produce the same series.

4. [20 points] Find the solution to the heat conduction problem

$$u_{xx} = 2u_t, \quad 0 < x < 1, \quad t > 0;$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0;$$

$$u(x,0) = \sin(2\pi x) + 2\sin(7\pi x).$$

Let $u = XT$

$$X''T = 2XT'$$

$$\frac{X''}{X} = \frac{2T'}{T} = -\sigma$$

$X'' + \sigma X = 0$ $\qquad$ $T' + \frac{1}{2}\sigma T = 0$

$X = \sin(\sqrt{\sigma}x)$ $\qquad$ $e^{-\frac{1}{2}\sigma t}$

$\sqrt{\sigma} = n\pi$

Let $u(x,t) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)\, e^{-\frac{n^2\pi^2}{2}t}$

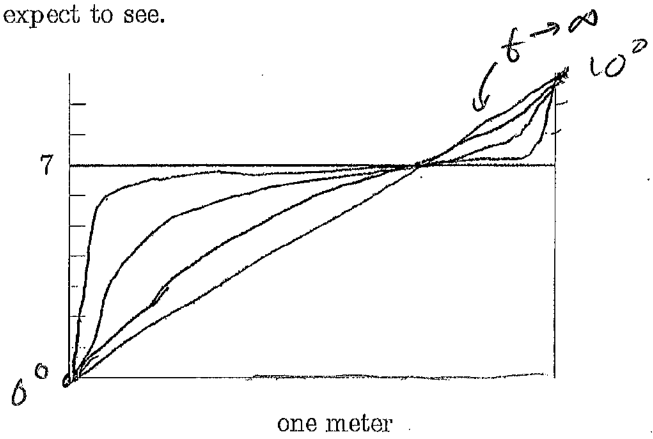
Now $u(x, 0) = \sum C_n \sin(n\pi x) = \sin(2\pi x) + 2\sin(7\pi x)$

$\Rightarrow C_2 = 1, \quad C_7 = 2$

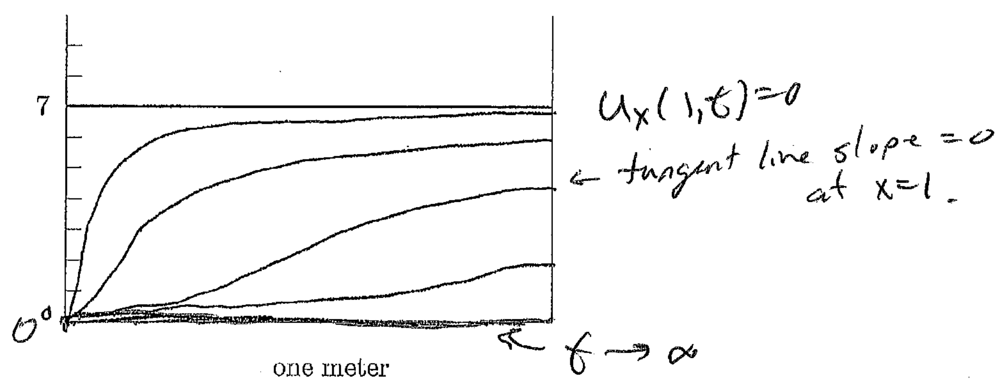
all others are zero.

$$u(x,t) = \left[\sin(2\pi x) + 2\sin(7\pi x)\right] e^{-\frac{n^2\pi^2}{2}t}.$$

5. [20 points] a. A one meter metal rod initially has uniform temperature of 7 C^o. The left end is then set to 0 C^o and the right end to 10 C^o. As $t \to \infty$ what will the temperature distribution approach? Graph this and sketch graphs for several intermediate temperature distributions we would expect to see.



b. A one meter metal rod initially has uniform temperature of 7 C^o. The left end is then set to 0 C^o while the left end is insulated. As $t \to \infty$ what will the temperature distribution approach? Graph this and sketch graphs for several intermediate temperature distributions we would expect to see.



"All the heat drains out" the left end.