2.3

#6 a. Find the velocity of a falling body after falling from height h. Let down be the positive direction, so $g = +9.8$. The object starts at $-h$ with 0 velocity. Thus

$$a = g$$
$$v = gt$$
$$s = \frac{gt^2}{2} - h$$

When is $s(t) = 0$? $t = \sqrt{\frac{2h}{g}}$. What is v then?

$$v\left(\sqrt{\frac{2h}{g}}\right) = g\sqrt{\frac{2h}{g}} = \sqrt{\frac{2g^2h}{g}} = \sqrt{2gh}.$$

b. Let V be the volume of liquid. Now

$$\frac{dV}{dt} = -\,(\text{the effective cross section}) \times (\text{the velocity of the flow})$$

$$= -\,\alpha a\sqrt{2gh}.$$

Now for a small time interval we can assume the tank cross section area $A(h)$ is fixed. Then $\Delta V \approx A(h)\Delta h$. Divide by Δt and take the limit as $\Delta t \to 0$ to get $\frac{dV}{dt} = A(h)\frac{dh}{dt}$.

Therefore,

$$A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh} \qquad (*)$$

2.3

#6 c. The tank is a cylinder. $A = \pi(1)^2 = \pi$.

Cross section of outlet is $a = \pi(.1)^2 = \pi/100$.

Recall $\alpha = 0.6$ and $g = 9.8$. Then by (*)

$$A \frac{dh}{dt} = -\alpha a \sqrt{2gh}$$

$$\int \frac{\cancel{\pi}\, dh}{\sqrt{h}} = \int -\alpha \frac{\cancel{\pi}}{100} \sqrt{2g}\, dt$$

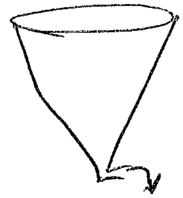
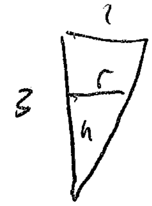
$$2\sqrt{h} = \underbrace{\frac{-\alpha}{100}\sqrt{2g}}_{\text{call this } b.\ \text{Then } b = -0.0265631.} t + C$$

When $t=0$, $h=3$. So $C = 2\sqrt{3}$.

When is $h=0$? Set $bt + c = 0$. Thus

$$t = -\frac{c}{b} = \frac{2\sqrt{3}}{0.0265631} = 130.4 \text{ seconds.}$$

When I first read the problem I thought the tank was a cone rather than a cylinder. Don't ask why. But I worked it out. Here is what I got. I assumed the vertex of the cone was down.

#6 C. With a cone!  

$\frac{r}{h} = \frac{1}{3} \Rightarrow r = \frac{h}{3}.$

Thus $A(h) = \pi r^2 = \frac{\pi}{9} h^2.$

Now $A(h)\frac{dh}{dt} = -\alpha a \sqrt{2gh}$ gives

$$\frac{\pi}{9} h^2 \frac{dh}{dt} = -\alpha \overset{\pi/100}{a} \sqrt{2g}\, h^{\frac{1}{2}}$$

$$h^{3/2}\, dh = \underbrace{-\frac{\alpha\, 9 \sqrt{2g}}{100}}_{= b = -0.2390681} dt$$

$$\int h^{3/2} dt = \int b\, dt$$

$$\frac{2}{5} h^{5/2} = bt + C.$$

At $t=0$ we know $h=3$. Thus $C = \frac{2}{5}(3)^{5/2} \approx 6.2353829$

When is $h=0$? Solve $bt + c = 0$.

$$t = \frac{-C}{b} = \frac{6.2353829}{0.2390681} \approx 26 \text{ seconds}$$

2.3

#24 a. $a = -\mu v^2$. $a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$.

Thus $v\frac{dv}{dx} = -\mu v^2 \Rightarrow \frac{dv}{dx} = -\mu v$.

Thus $v = Ce^{-\mu x}$. At $x=0$ $v = 150$ mph, so $C = 150$.

$$\boxed{v = 150e^{-\mu x}}$$

b. When $x = 2000$ ft, $v = 15$ mph. 2000ft $= \frac{2000}{5280}$ miles.

$$15 = 150e^{-\mu x}$$

$$\frac{1}{10} = e^{-\mu x}$$

$$-\mu x = \ln\frac{1}{10} = -\ln 10$$

$$\mu = \frac{\ln 10}{x} = 6.0788246$$

c. Now we need v in terms of time.

$$\frac{dv}{dt} = a = -\mu v^2$$

$$\int \frac{1}{v^2}\,dv = \int -\mu\,dt.$$

$$-\frac{1}{v} = -\mu t + C$$

At $t=0$, $v = 150$. Thus $C = -1/150$. Now let $v = 15$ and find t. $-\frac{1}{15} = -\mu t + \frac{-1}{150} \Rightarrow t = \frac{1}{\mu}\left(\frac{9}{150}\right) \approx 0.0098703$ hours

≈ 35.5 seconds.

2.3

#29 a. $V_0 = \sqrt{2gR}$, $R = 4000$ miles. $g = 32\ ft/s^2$.

By eq (30) in the text $V = \sqrt{\cancel{2g^2R^2} - 2gR + \dfrac{2gR^2}{R+x}}$.

Let $x = 240,000$ miles. Convert g to miles/hour^2

$$g = 32\ \frac{3600}{5280} \simeq 21.81\overline{81}$$

Then $V = 53.49206384$ m/hr.

b. By eq (26) $m\dfrac{dv}{dt} = -\dfrac{mgR^2}{(R+x)^2}$.

2.3

#29 a. $V_0 = \sqrt{2gR}$. $R = 4000$ miles. $g = 32\ ft/s^2 = \frac{32 \times 3600}{5280}$ miles/h^2

$= 21.81$.

By Eq (30) in Example 4

$$v = \sqrt{2gR - 2gR + \frac{2gR^2}{R+x}} = R\sqrt{\frac{2g}{R+x}} \quad (*)$$

Let $x = 240{,}000$ miles. Then $v(x) = 53.49$ mi/h.

b. We want to relate time t and v or x.
Rewrite $(*)$ as

$$\frac{dx}{dt} = R\sqrt{\frac{2g}{R+x}}$$

$$\int \sqrt{R+x}\, dx = R\sqrt{2g}\int dt = R\sqrt{2g}\, t + C$$

$$\frac{2}{3}(R+x)^{3/2} = R\sqrt{2g}\, t + C \qquad \text{This relates } x \text{ and } t.$$

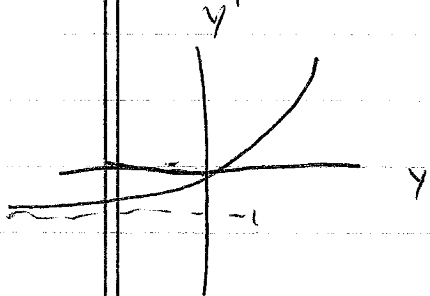
When $t = 0$, $x = 0$ so $C = \frac{2}{3}R^{3/2}$.

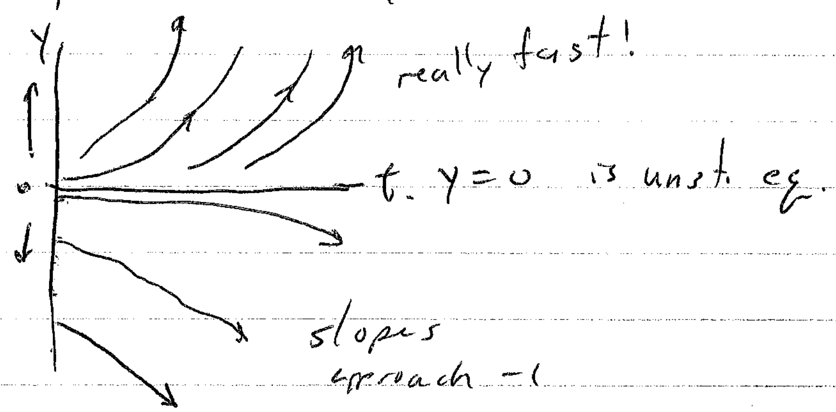
Then $$t = \frac{2}{3}\,\frac{(R+x)^{3/2} - R^{3/2}}{R\sqrt{2g}}$$

At $x = 249{,}000$, I get $t = 3034.5667$ hours.
This is way off from what the book gets,
but I haven't found a mistake. I'll
give you five bonus points on the
test if you can find it, on your own.

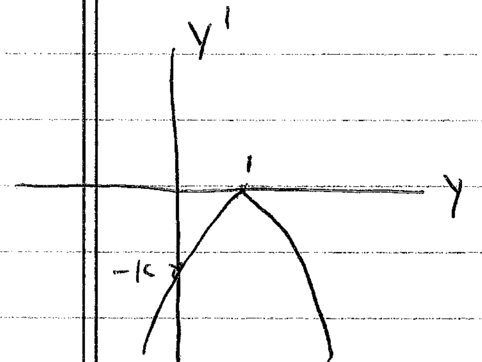
2.4 #33 See problem 2 on Quiz 2.

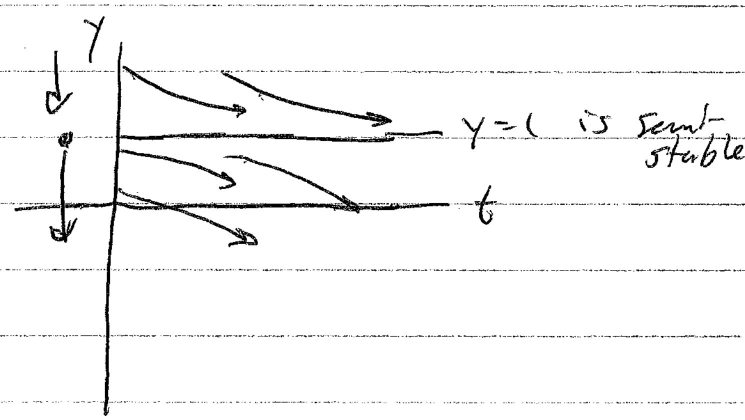
2.5 #4 $y' = e^y - 1$ $\qquad$ $y' = 0$ when $y = 0$.



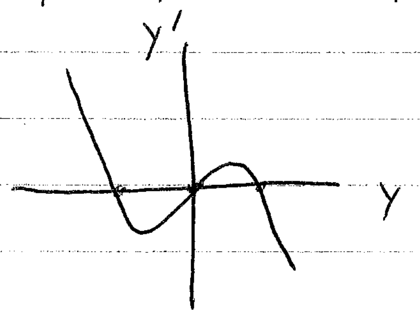


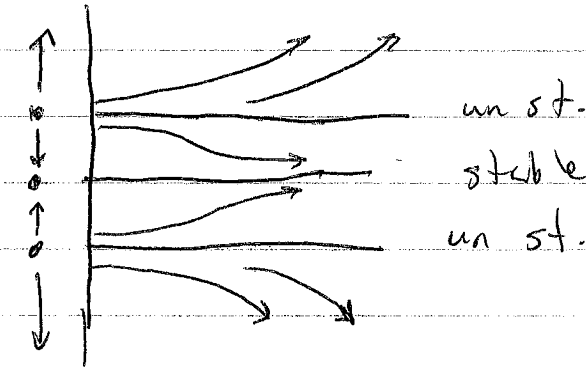
#8 $y' = -k(y-1)^2$ $\qquad$ $k > 0$





#10 $y' = y(1-y^2) = y(1-y)(1+y)$





2.5

21 $y' = r(1 - \frac{y}{K})y - h \qquad r, K, h > 0.$

(a) Assume $h < rK/4$. Show $\exists$ two eq. solutions.

$$y' = ry - \frac{r}{K}y^2 - h = -\frac{r}{K}y^2 + ry - h = 0$$

$$y = \frac{-r \pm \sqrt{r^2 - 4\frac{r}{K}h}}{-2r/K} \quad (*)$$

$$h < \frac{rK}{4} \Rightarrow \frac{4h}{K} < r \Rightarrow \frac{4hr}{K} < r^2 \Rightarrow 0 < r^2 - \frac{4rh}{K}.$$

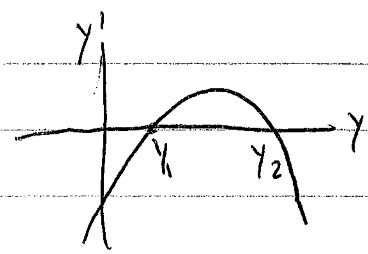
Thus there are two eq. solutions given by $(*)$.

Let $y_1 = \frac{rK - \sqrt{r^2K^2 - 4rKh}}{2r} = \frac{K}{2} - \sqrt{\frac{K^2}{4} - \frac{Kh}{r}}$

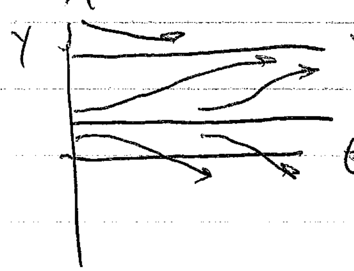
Let $y_2 =$ " $= \frac{K}{2} + \sqrt{\text{same}}$

Then $y_1 < y_2$. Also $0 < y_1$.

(b) $r(1 - \frac{y}{K})y - h$ is a downward opening parabola.



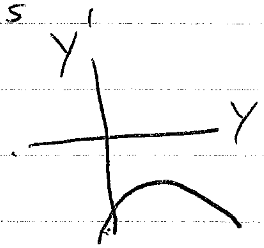
Hence



2.5 #21

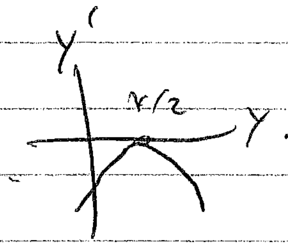
(c) The conclusion follows from the graph above.

(d) If $h > \frac{rK}{4}$, there are no real roots, and the y' vs y graph looks like

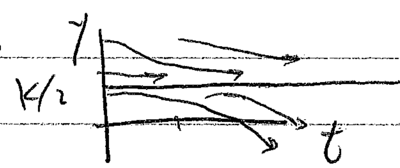


Thus y is always decreasing and there are no eq. solutions. Once $y=0$ is reached there are no more fish and the model is no longer relevent.

(e) If $h = \frac{rK}{4}$, there is only one root and the y' vs y graph looks like



Thus the y vs t ~~graph~~ solution are as shown here →

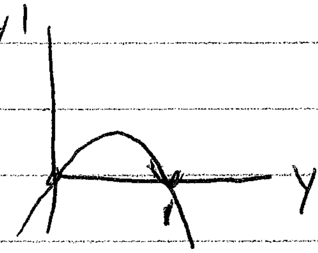


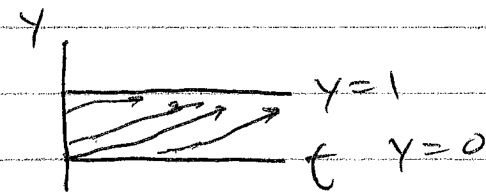
$$y = \frac{-r \pm \sqrt{0}}{-2r/K} = \frac{K}{2}$$

2.5
#22

$y' = \alpha y(1-y) \qquad y(0) = y_0. \qquad \alpha > 0.$

(a) Eq points are $y=0$ and $y=1$.





(b)

$$\int \frac{dy}{y(1-y)} = \int \alpha \, dt = \alpha t + C$$

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y} \qquad \begin{matrix} -A+B=0 \\ A=1,\ B=1 \end{matrix}$$

$$\int \frac{1}{y} + \frac{1}{1-y} \, dy = \ln y - \ln 1-y = \alpha t + C.$$

$$\ln\left(\frac{y}{1-y}\right) = \alpha t + c$$

$$\frac{y}{1-y} = Ce^{\alpha t} \qquad C = \frac{y_0}{1-y_0}$$

$$y + yCe^{\alpha t} = Ce^{\alpha t}$$

$$y = \frac{Ce^{\alpha t}}{1+Ce^{\alpha t}} = \frac{C}{e^{-\alpha t}+C} \rightarrow \frac{C}{+C} = 1.$$

limit as $t \to \infty$.