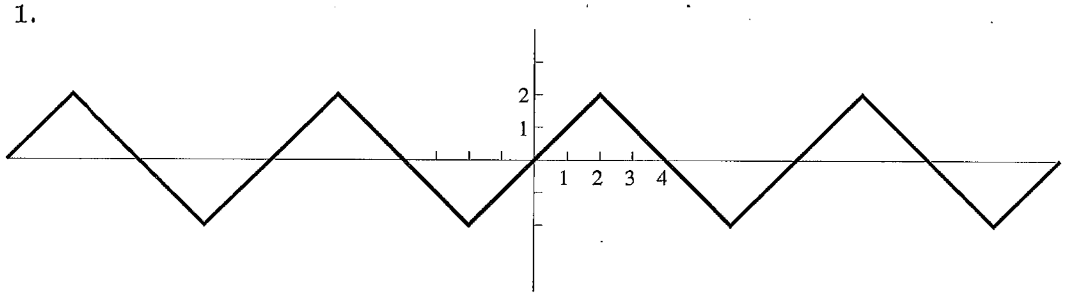
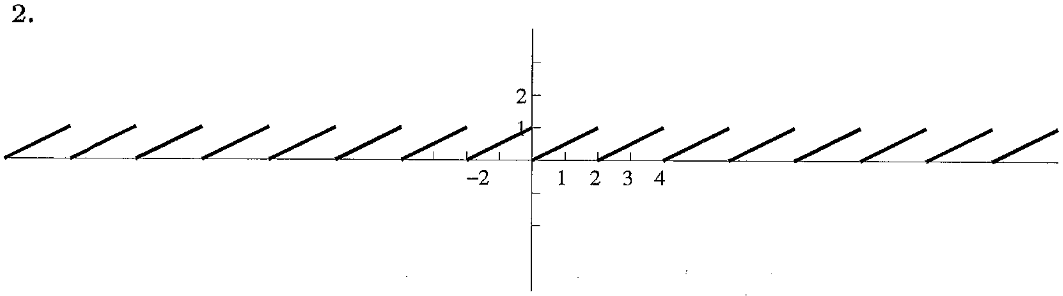
Find the Fourier Series for each of the functions shown below.

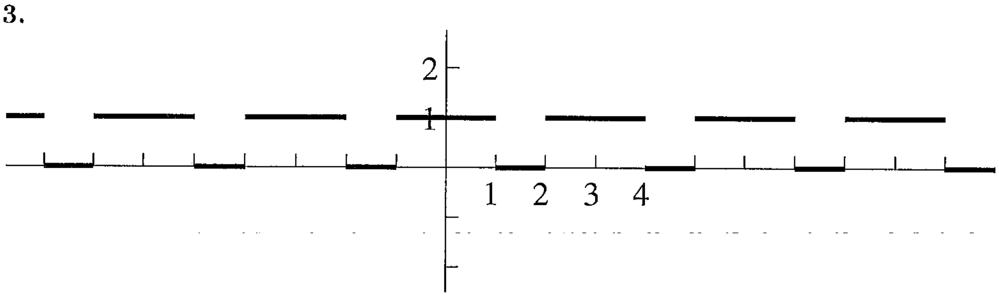
1.



2.



3.



Extra Credit. Pick one of the above and use a computer to plot the partial sums for $n = 1, 2, 3, 5, 10, \& 100$.

Scroll down for solutions!

1. Let $f(x) = \begin{cases} -x-4 & x \in [-4,-2] \\ x & x \in [-2,2] \\ -x+4 & x \in [2,4] \end{cases}$, and then

extend periodically. The period is 8 so $L=4$.
The function is odd so we will only need the sine terms, ie. $a_n = 0$ for $n \geq 0$.

$$b_n = \frac{1}{4}\int_{-4}^{4} \sin\left(\frac{n\pi x}{4}\right) f(x)\,dx = \frac{1}{2}\int_0^4 \sin\left(\frac{n\pi x}{4}\right) f(x)\,dx. \text{ (why?)}$$

$$b_n = \frac{1}{2}\int_0^2 \sin\left(\frac{n\pi x}{4}\right) x\,dx + \frac{1}{2}\int_2^4 \sin\left(\frac{n\pi x}{4}\right)(-x+4)\,dx$$

$$\int x\sin(\alpha x)\,dx = ? \qquad \left[\alpha = \frac{n\pi}{4}\right].$$

Let $u = x$ and $dv = \sin(\alpha x)\,dx$.
Then $du = dx$ and $v = \frac{-1}{\alpha}\cos(\alpha x)$.
Thus $? = -\frac{x}{\alpha}\cos(\alpha x) + \frac{1}{\alpha}\int \cos(\alpha x)\,dx$

$$= -\frac{x}{\alpha}\cos(\alpha x) + \frac{1}{\alpha^2}\sin(\alpha x) + C.$$

Thus,

$$b_n = \frac{1}{2}\left[-\frac{x}{\alpha}\cos(\alpha x) + \frac{1}{\alpha^2}\sin(\alpha x)\right]_0^2 - \frac{1}{2}\left[\text{ same }\right]_2^4$$

$$+ 2\int_2^4 \sin(\alpha x)\,dx \rightarrow -\frac{2}{\alpha}\cos(\alpha x)\Big|_2^4$$

$$= \frac{1}{2}[\ * \ ] - \frac{1}{2}[\ 0\ ] - \frac{1}{2}[\ ** \ ] + \frac{1}{2}[\ * \ ] + \frac{-2}{\alpha}\cos(n\pi) + \frac{2}{\alpha}\cos\left(\frac{n\pi}{2}\right)$$

$$= \left[\frac{-8}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right] - \frac{1}{2}\left(\frac{-16}{n\pi}\cos(n\pi) - \frac{16}{n^2\pi^2}\underbrace{\sin(n\pi)}_{=0}\right)$$

$$-\frac{8}{n\pi}\cos(n\pi) + \frac{8}{n\pi}\cos\left(\frac{n\pi}{2}\right) = \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).$$

Thus, $b_n = \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$. Now I'll make a table and see the pattern.

$b_1 = 16/\pi^2$

$b_2 = 0$

$b_3 = \frac{-16}{9\pi^2}$

$b_4 = 0$

$b_5 = \frac{+16}{25\pi^2}$

$b_6 = 0$, all even n give $b_n = 0$.

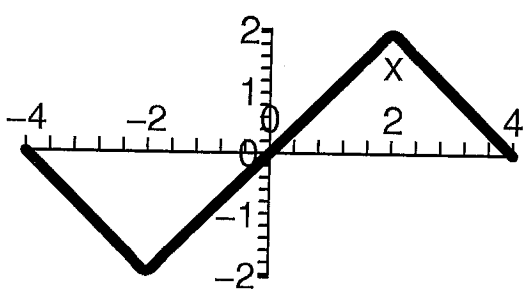
Let $n = 2k-1$. Then the Fourier series is

$$\sum_{k=1}^{\infty} \frac{16(-1)^{k+1}}{(2k-1)^2\pi^2}\sin\left(\frac{(2k-1)\pi x}{4}\right)$$

```
> FS:= x->16/Pi^2*sum((-1)^(k+1)*sin((2*k-1)*Pi*x/4)/(2*k-1)^2,k=1..10);
```

$$FS := x \to \frac{16\left(\sum_{k=1}^{10} \frac{(-1)^{(k+1)}\sin\left(\frac{1}{4}(2k-1)\pi x\right)}{(2k-1)^2}\right)}{\pi^2}$$

```
> plot(FS(x),x=-4..4,color=black,thickness=3);
```



2. The period is 2, so $L=1$. So let

$$f(x) = \begin{cases} \frac{1}{2}x+1 & x \in [-1,0) \\ \frac{1}{2}x & x \in (0,1] \end{cases}, \text{ and extend peridically.}$$

It is not even or odd. So we need to find the a_n's and b_n's. The ave. value of $f(x)$ is $\frac{1}{2}$. So $a_0 = 1$. Notice if you subtract $\frac{1}{2}$ from $f(x)$ the resulting function is odd (do this graphically) so all the other a_n's must be zero! We will check this anyway.

$$a_n = \frac{1}{1}\int_{-1}^{1} \cos(n\pi x) f(x)\,dx =$$

$$\int_{-1}^{0} \cos(n\pi x)\cdot\frac{1}{2}x\,dx + \int_{-1}^{0} \cos(n\pi x)\cdot 1\,dx + \int_{0}^{1} \cos(n\pi x)\cdot\frac{1}{2}x\,dx$$

$$= \underbrace{\int_{-1}^{1} \cos(n\pi x)\frac{1}{2}x\,dx}_{\text{odd function!}} + \int_{-1}^{0} \cos(n\pi x)\,dx$$

$$= 0 + \frac{1}{n\pi}\sin(n\pi x)\Big|_{-1}^{0} = 0. \quad \text{See!}$$

$$b_n = \cdots = \int_{-1}^{1} \sin(n\pi x)\frac{1}{2}x\,dx + \int_{-1}^{0} \sin(n\pi x)\,dx =$$

$$\frac{1}{2}\left[\frac{-x}{n\pi}\cos(n\pi x) + \frac{1}{n^2\pi^2}\underbrace{\sin(n\pi x)}_{0}\right]_{-1}^{1} - \frac{1}{n\pi}\cos(n\pi x)\Big|_{-1}^{0}$$

$$\frac{1}{2}\left[\frac{-1}{n\pi}\cos(n\pi) - \frac{1}{n\pi}\underset{\cos(n\pi)}{\underset{\parallel}{\cos(-n\pi)}}\right] - \left(\frac{1}{n\pi} - \frac{1}{n\pi}\underset{\cos(n\pi)}{\underset{\parallel}{\cos(-n\pi)}}\right)$$

$$= \underline{\frac{-1}{n\pi}\cos(n\pi)} - \frac{1}{n\pi} + \underline{\frac{1}{n\pi}\cos(n\pi)}$$

$$= -1/n\pi$$

~~Weha~~

We have $b_n = \frac{-1}{n\pi}$, $n = 1, 2, 3, \dots$

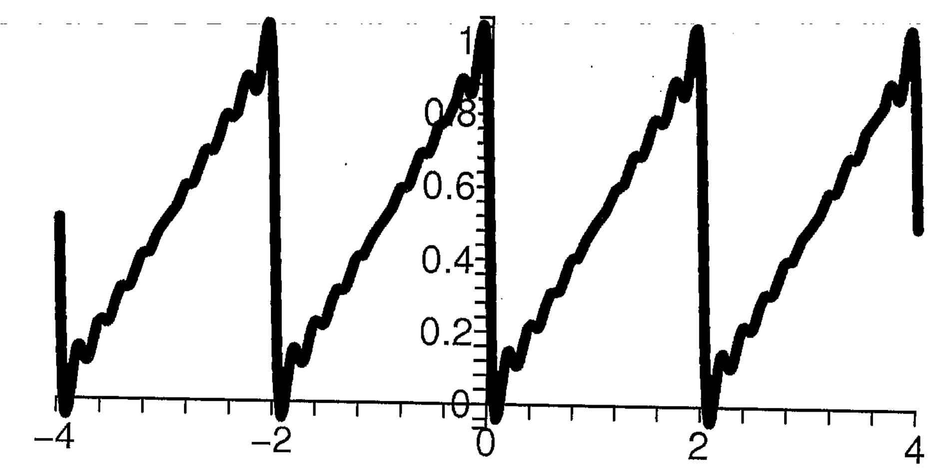
and $a_0 = 1$. Thus the Fourier Series of $f(x)$ is

$$\frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n\pi}.$$

```
> FS2:= x-> 1/2-sum(1/n/Pi*sin(n*Pi*x),n=1..10);
```

$$FS2 := x \rightarrow \frac{1}{2} - \left(\sum_{n=1}^{10} \frac{\sin(n\,\pi\,x)}{n\,\pi} \right)$$

```
> plot(FS2(x),x=-4..4,color=black,thickness=3);
```



3. The period is 3, so $L = 1.5$. Let

$$f(x) = \begin{cases} 0 & x \in (-1.5, -1) \\ 1 & x \in (-1, 1) \\ 0 & x \in (1, 1.5] \end{cases}$$, extend periodically.

It is even, so all the b_n's are zero. The ave. value is $2/3$ so $a_0 = 4/3$.

$$a_n = \frac{1}{1.5}\int_{-1.5}^{1.5} \cos\left(\frac{n\pi x}{1.5}\right) f(x)\,dx = \frac{2}{3}\int_{-1}^{1} \cos\left(\frac{2n\pi x}{3}\right) dx$$

even

$$= \frac{4}{3}\int_0^1 \cos\left(\frac{2n\pi x}{3}\right) dx = \frac{4}{3}\cdot\frac{3}{2n\pi}\cdot \sin\left(\frac{2n\pi x}{3}\right)\Big|_0^1$$

$$= \frac{2}{n\pi}\sin\left(\frac{2n\pi}{3}\right).$$ Notice every 3^{rd} term will be zero, and that the func. is zero $1/3^{rd}$ of the time!

$a_1 = \frac{2}{\pi}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{\pi}$

$a_2 = \frac{2}{2\pi}\cdot\frac{-\sqrt{3}}{2} = \frac{-\sqrt{3}}{2\pi}$

$a_3 = 0$

$a_4 = \frac{1}{4}\frac{\sqrt{3}}{\pi}$

$a_5 = -\frac{1}{5}\sqrt{3}/\pi$

$a_6 = 0$

etc.

Let $$Z(n) = \begin{cases} 1 & n \bmod 3 = 1 \\ -1 & n \bmod 3 = 2 \\ 0 & n \bmod 3 = 0 \end{cases}$$

Then the Fourier Series is

$$\frac{2}{3} + \sum_{n=1}^{\infty} \frac{Z(n)\sqrt{3}}{n\pi}\cos\left(\frac{2n\pi x}{3}\right)$$

```
> z:=proc(n) # Do you see what this procedure does?
   if n mod 3 = 1 then 1.0 else if n mod 3 = 2 then -1.0 else 0.0 fi fi
  end proc;
```

```
z := proc(n)
    if `mod`(n, 3) = 1 then
        1.0
    else
        if `mod`(n, 3) = 2 then -1.0 else 0. end if;
    end if;
end proc;
```

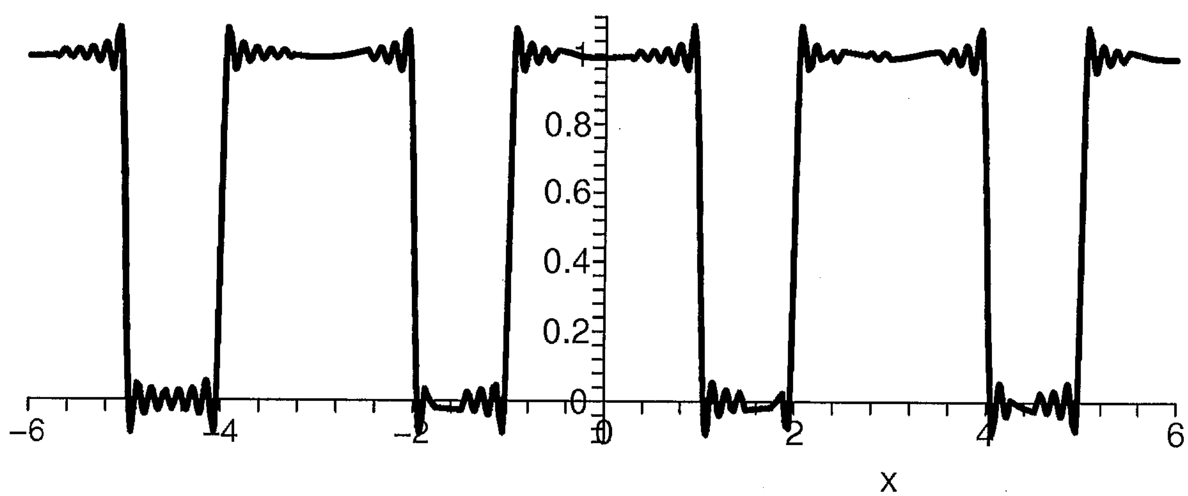
```
> z(1);z(2);z(3);z(4);z(5);z(6);
```

$$1.0$$
$$-1.0$$
$$0.$$
$$1.0$$
$$-1.0$$
$$0.$$

```
> FS3:= x->2/3+sqrt(3)/Pi*add(z(n)*cos(2*n*Pi*x/3)/n,n=1..20);  # I had to
  use "add" because "sum" didn't work with a proc input. Weird.
```

$$FS3 := x \to \frac{2}{3} + \frac{\sqrt{3}\ add\left(\frac{z(n)\cos\left(\frac{2}{3}\, n\, \pi\, x\right)}{n},\ n = 1\,..\,20\right)}{\pi}$$

```
> plot(FS3(x),x=-6..6,color=black,thickness=2);
```




```
>
```