Math 305 Test 2 Fall 2012

Name: ______________________ Section time: ______________________

SCIENTIFIC CALCULATORS ALLOWED

1. [15 points] Find the general solution to each of the following.

a. $y'' - y' = 6y$

$y'' - y' - 6y = 0$

$r^2 - r - 6 = 0$

$(r-3)(r+2) = 0 \quad r = 3, -2$

$y(t) = C_1 e^{3t} + C_2 e^{-2t}$

b. $y'' + 2y' + 5y = 0$

$r^2 + 2r + 5 = 0$

$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 5}}{2} = -1 \pm \sqrt{-4} = -1 \pm 2i$

$y(t) = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)$

c. $t^2y'' - 4ty' + 6y = 0$ for $(t > 0)$. Hint: Try $y = t^n$.

$y' = nt^{n-1}$

$y'' = n(n-1)t^{n-2}$

$n(n-1)t^2 - 4nt^n + 6t^n = 0$

$(n^2 - n - 4n + 6)\, t^n = 0 \quad (t > 0)$

$n^2 - 5n + 6 = 0$

$(n-3)(n-2) = 0 \quad n = 3, 2$

$y(t) = C_1 t^2 + C_2 t^3$

2. [10 points] Find the general solution to $y'' + 2y' + y = 2e^{-t}$.

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1$

$y_h = C_1 e^{-t} + C_2 t e^{-t}$

Try $y_p = At^2e^{-t}$

$y_p' = 2Ate^{-t} - At^2e^{-t}$

$y_p'' = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$

plug in

$[(2A - 4At + At^2) + (4At - 2At^2) + At^2]e^{-t} = 2e^{-t}$

$\underset{0}{(A - 2A + A)}t^2 + \underset{0}{(-4A + 4A)}t + 2A = 2$

$2A = 2$

$A = 1$

$y = y_h + y_p = C_1 e^{-t} + C_2 t e^{-t} + t^2 e^{-t}$

3. [20 points] For each pair of functions below determine if it is linearly dependent or independent with respect to the given interval. (Only find the Wronskian when you need to.)

a. $\{\ln(x+1), \ln(x^2+2x+1)\}$; $(-1, \infty)$

$\ln(x+1)^2 = 2\ln(x+1)$ Thus L.D. since one is a multiple of the other.

b. $\{x^2|x|, x^3\}$; $(-1, 1)$ $\quad f\left(\frac{-1}{2}\right) = \frac{1}{8} \quad g\left(\frac{-1}{2}\right) = -\frac{1}{8}$

$C_1 f(x) + C_2 g(x) = 0 \qquad \frac{C_1}{8} - \frac{C_2}{8} = 0 \Rightarrow C_1 = C_2$

$f\left(\frac{1}{2}\right) = \frac{1}{8} \quad g\left(\frac{1}{2}\right) = \frac{1}{8}.$

$\frac{C_1}{8} + \frac{C_2}{8} = 0 \Rightarrow C_1 = -C_2$

Hence $C_1 = C_2 = 0$ is only solution. Thus L.I.

c. $\{x^2|x|, x^3\}$; $(-1, 0)$

If $x < 0$, $|x| = -x$. Thus $f(x) = x^2|x| = -x^3 = -g(x)$.

Hence L.D. since one is a mult. of the other on $(-1,0)$.

d. $\{t\sin t, \sin t\}$; $(-\infty, \infty)$

$$\begin{vmatrix} t\sin t & \sin t \\ \sin t + t\cos t & \cos t \end{vmatrix} = tsc - s^2 - tsc = -\sin^2(x).$$

This not always 0, hence $t\sin t$ & $\sin t$ are L.I.

4. [5+20 points] Consider the equation

$$t^2y'' - t(t+2)y' + (t+2)y = 0$$

for $t > 0$.

a. Show that $y_1(t) = t$ is a solution.

$Y_1' = 1 \quad Y_1'' = 0$

$t^2 \cdot 0 - t(t+2)\cdot 1 + (t+2)t = 0$ ✓

b. Let $y_2(t) = v(t) \cdot t$. Substitute $y_2(t)$ into the given equation and find a function $v(t)$ so that $y_2(t)$ is a second linearly independent solution.

$y = vt$

$y' = v't + v$

$y'' = v''t + v' + v'$

$= v''t + 2v'$

plug in.

$t^2(v''t + 2v') - t(t+2)(v't + v) + (t+2)vt = 0$

$t^3v'' + 2t^2v' - \cancel{t}(t+2)[v't^2 + \cancel{vt} - \cancel{vt}] = 0$

$t^3v'' + \cancel{2t^2v'} - t^3v' - \cancel{2t^2v'} = 0$

$t^3v'' - t^3v' = 0$

$v'' = v'$

Let $w = v'$.

$w' = w$

$w = e^t$ will work.

$v = \int e^t dt = e^t$

$\boxed{Y_2 = te^t}$

5. [20 points] A mass weighing 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with damping constant 2 lb-s/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 3 in/s find its position u as a function of time t. (Watch your units.) Express your answer in the form $u(t) = Re^{at}\cos(\omega t - \delta)$.

Step 1. Find the mass m and the spring constant k.

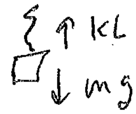
Step 2. Write down then differential equation for this system.

Step 3. Find the general solution.

Step 4. What are the initial conditions?

Step 5. Use these to find the coefficents.

Step 6. Convert to the form $u(t) = Re^{at}\cos(\omega t - \delta)$



$mg = 16$ $\qquad kL = mg \qquad \gamma = 2$ given

$m = \frac{16}{g} = \frac{16}{32} = \frac{1}{2} \qquad k = \frac{mg}{.25} = \frac{16}{\frac{1}{4}} = 64$

$\frac{1}{2}u'' + 2u' + 64u = 0$

$u = \frac{-2 \pm \sqrt{2^2 - 128}}{1} = -2 \pm \sqrt{-124} = -2 \pm 2\sqrt{31}\,i$

$u(t) = C_1 e^{-2t}\cos(2\sqrt{31}\,t) + C_2 e^{-2t}\sin(2\sqrt{31}\,t)$

$u(0) = 0 \qquad u'(0) = .25 \qquad u(0) = C_1 \Rightarrow C_1 = 0.$

$u' = -2C_2 e^{-2t}\sin(\ ) + C_2 e^{-2t}\, 2\sqrt{31}\cos(\ )$

$u'(0) = 0 + 2\sqrt{31}\, C_2 = \frac{1}{4}$

$C_2 = \frac{1}{8\sqrt{31}}$

$R = \sqrt{C_1^2 + C_2^2} = \sqrt{0 + C_2^2} = C_2 = \frac{1}{8\sqrt{31}} \qquad \delta = \tan^{-1}\left(\frac{C_2}{C_1}\right) = \tan^{-1}(\text{"}\infty\text{"}) = \frac{\pi}{2}$

$$\boxed{u(t) = \frac{1}{8\sqrt{31}}\cos\left(2\sqrt{31}\,t - \frac{\pi}{2}\right)}$$

6. [10 points] Suppose the $y = C_1 \sin^2(t) + C_2 \cos^2(t)$ is the general solution to $y'' + p(t)y' + q(t)y = 0$. What can we conclude about $p(t)$? Hint: Recall Abel's formula

$$W(y_1, y_2)(t) = Ce^{-\int p(t)\,dt}.$$

$$W(\ ,\ ) = \begin{vmatrix} \sin^2 t & \cos^2 t \\ 2sc & -2sc \end{vmatrix} = -2s^3c - 2sc^3$$
$$= -2sc(s^2+c^2)$$
$$= -2sc = -\sin(2t).$$

This is zero for $t = n\frac{\pi}{2}$. Hence $p(t)$ is not continuous at these points.

7. [10 BONUS points] Suppose we know that $y_1(t) = t$ and $y_2(t) = \sin t$ form a fundamental solution set for

$$y'' + p(t)y' + q(t)y = 0.$$

determine $p(t)$ and $q(t)$. Hint: just plug in and back-solve.

$\underline{y_1' = 1 \quad y_2'' = 0}$ $\quad y' = \cos t \quad y'' = -\sin t$

$p \cdot 1 + qt = 0$

$p = -qt$

$-\sin t + p\cos t + q\sin t = 0$

$-qt\cos t + q\sin t = \sin t$

$$q = \frac{\sin t}{\sin t - t\cos t}.$$

$$p = tq = \frac{t\sin t}{\sin t - t\cos t}$$