Math 305 Test 1 Fall 2012

Name: ______________ Section time: ______________

ONLY NON GRAPHING CALCULATORS ALLOWED

1. [10 points] a. Solve $\frac{dy}{dt} = y(K-y)$ where $K > 0$ and $y(0) = y_0 > 0$. Find y as a function of t.
   b. What in $\lim_{t\to\infty} y(t)$?

a $\int \frac{1}{y(k-y)}\,dy = \int dt = t + c$

$\frac{1}{y(k-y)} = \frac{A}{y} + \frac{B}{k-y}$ $\quad A=B$, $kA=1$ $\quad = \int \frac{1}{y} + \frac{1}{k-y}\,dy = \ln|y| + \ln|k-y| = \ln\left|\frac{y}{k-y}\right|$

Thus $\left|\frac{y}{k-y}\right| = e^{t+c} = e^c e^t$ so $\frac{y}{k-y} = \pm e^c e^t = Ce^t$ ☆ (Let $C = \pm e^c$).

Solve for y.

$y = (k-y)Ce^t = kCe^t - yCe^t$

$y + yCe^t = kCe^t$ $\quad$ At $t=0$, $y=y_0$. Use ☆

$y = \frac{kCe^t}{1+Ce^t}$ $\quad C = \frac{y_0}{k-y_0}$

$y = \frac{kC}{e^{-t}+C} = \frac{k\frac{y_0}{k-y}}{e^{-t}+\frac{y_0}{k-y}} = \frac{y_0 k}{(k-y)e^{-t}+y_0}$

b. $\lim_{t\to\infty} y(t) = \frac{y_0 k}{0+y_0} = k.$

2. [10 points] Find the general solution to

$$\frac{dy}{dx} = \frac{y}{x} + \csc\left(\frac{y}{x}\right).$$

You do not need to solve for y or x buy may leave your answer as a relation.

Let $v = \frac{y}{x}$. Then $y = xv$.

Thus $\frac{dy}{dx} = v + x\frac{dv}{dx} = v + xv'$

So,

$$v + xv' = v + \csc(v)$$

$$x v' = \csc(v)$$

$$\int \sin(v)\, dv = \int \frac{1}{x}\, dx$$

$$-\cos(v) = \ln|x| + c$$

$$\cos\left(\frac{y}{x}\right) = \ln\left(\frac{1}{|x|}\right) + c$$

3. [10 points] a. Show that

$$\underbrace{(3x^3y^2 + 2x^2y)}_{M} + \underbrace{(x^4y + 2xy^2)}_{N}y' = 0$$

is not exact.

$M_y = 6x^3y + 2x^2 \qquad N_x = 3x^3y + 2y^2. \qquad M_y \neq N_x$

So not exact.

b. Multiply through by $1/xy$ and show the new equation is exact.

$$\underbrace{(3x^2y + 2x)}_{M} + \underbrace{(x^3 + 2y^2)}_{N}y' = 0$$

$M_y = 3x^2 + 0 \qquad N_x = 3x^2 + 0 \qquad M_y = N_x$

so exact!

c. Find the general solution. You may leave your answer as a relation.

$\psi = \int M\,dx = \int 3x^2y + 2x\,dx = x^3y + x^2 + C_1(y)$

$\psi_y = \int N\,dy = \int x^3 + 2y^2\,dy = x^3y + y^3 + C_2(x).$

Let $\psi = x^3y + x^2 + y^3$

general solutoon is $x^3y + x^2 + y^3 = C.$

4. [10 points] Find the general solution to

$$y' + t^2 y = t^2 y^4.$$

Hint: It is a Bernoulli type equation.

$n=4$ Let $v = y^{1-n} = y^{1-4} = y^{-3}$.

Thus $y = v^{-\frac{1}{3}}$. So $y' = -\frac{1}{3} v^{-\frac{4}{3}} v'$

$$-\frac{1}{3} v^{-\frac{4}{3}} v' + t^2 v^{-\frac{1}{3}} = t^2 v^{-\frac{4}{3}}$$

$$-\frac{1}{3} v' + t^2 v = t^2$$

$$v' - 3t^2 v = -3t^2$$

$$\mu = e^{\int -3t^2 dt} = e^{-t^3}.$$

$$e^{-t^3} v' - 3t^2 e^{-t^3} v = -3t^2 e^{-t^3}$$

$$(e^{-t^3} v)' = -3t^2 e^{-t^3}$$

$$e^{-t^3} v = \int -3t^2 e^{-t^3} dt$$ Let $u = -t^3$, $du = -3t^2 dt$

$$= \int e^u du$$

$$e^{-t^3} v = e^u + C = e^{-t^3} + C$$

$$v = 1 + C e^{t^3}$$

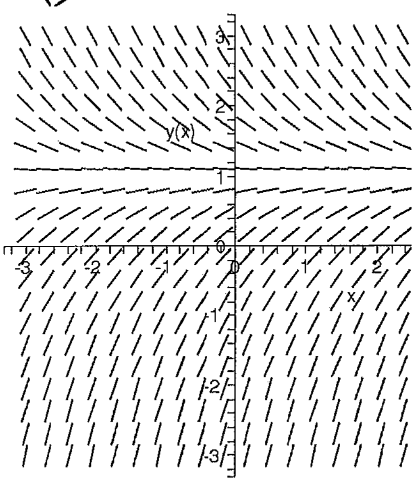
$$v = \frac{1}{y^3}$$

So $y = \sqrt[3]{\dfrac{1}{1 + Ce^{t^3}}}$.

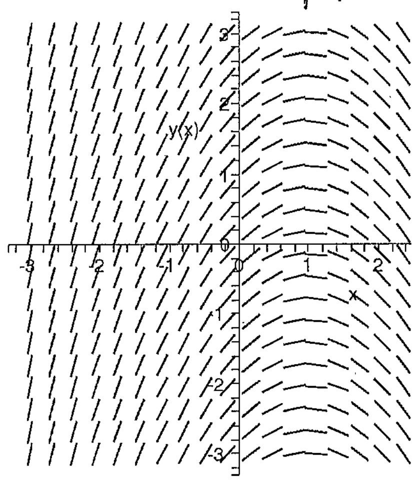
5. [10 points] You are in the middle of giving a presentation as part of a job interview. Your roommate, always the prankster, has mixed up your direction field slides an inserted one your weren't going to use. Match the differential equations with their corresponding direction fields. (Each correct match is worth 2 points, each incorrect match is -1 point.)

(a) $y' = 1 - x$
(b) $y' = x - y^2$
(c) $y' = (1 - y)^2$
(d) $y' = 1 - y^2$
(e) $y' = 1 - y$

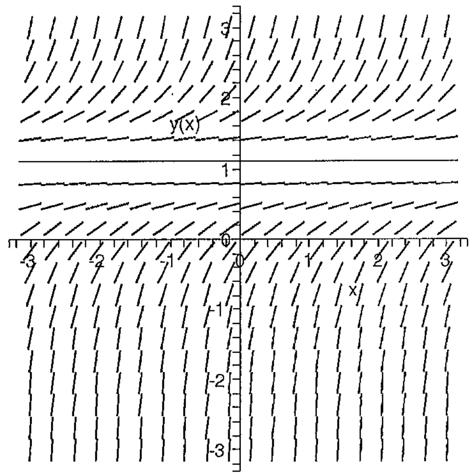
E



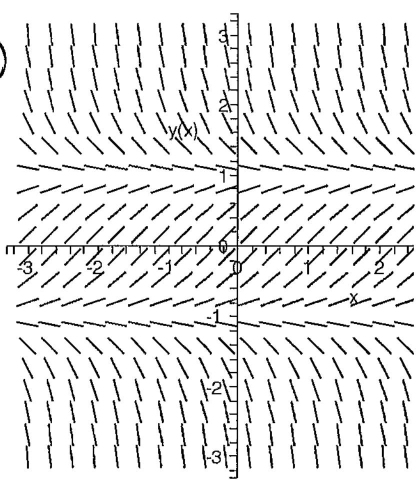
A



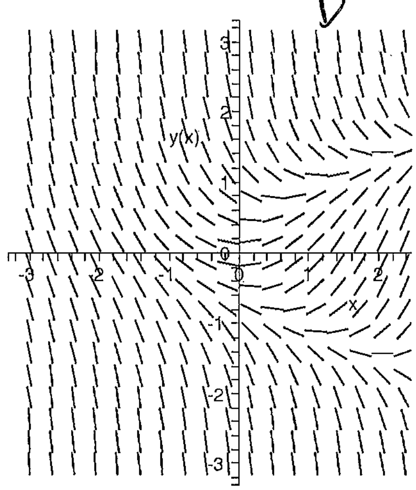
C

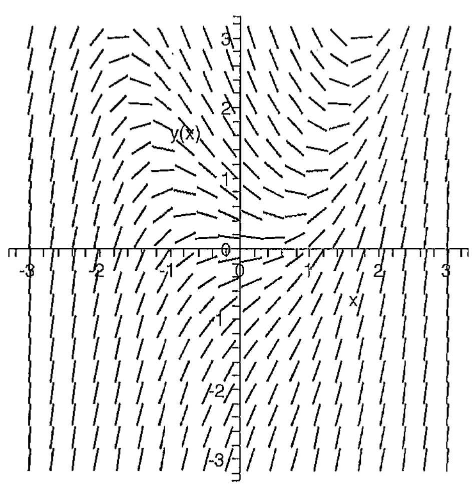


D

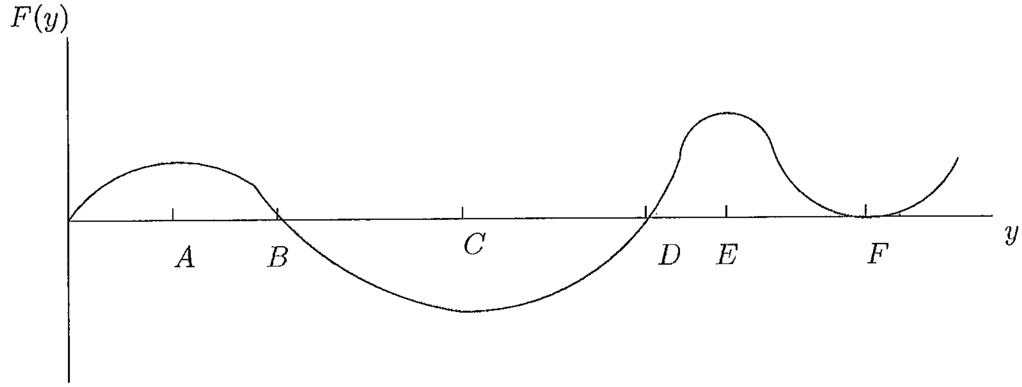


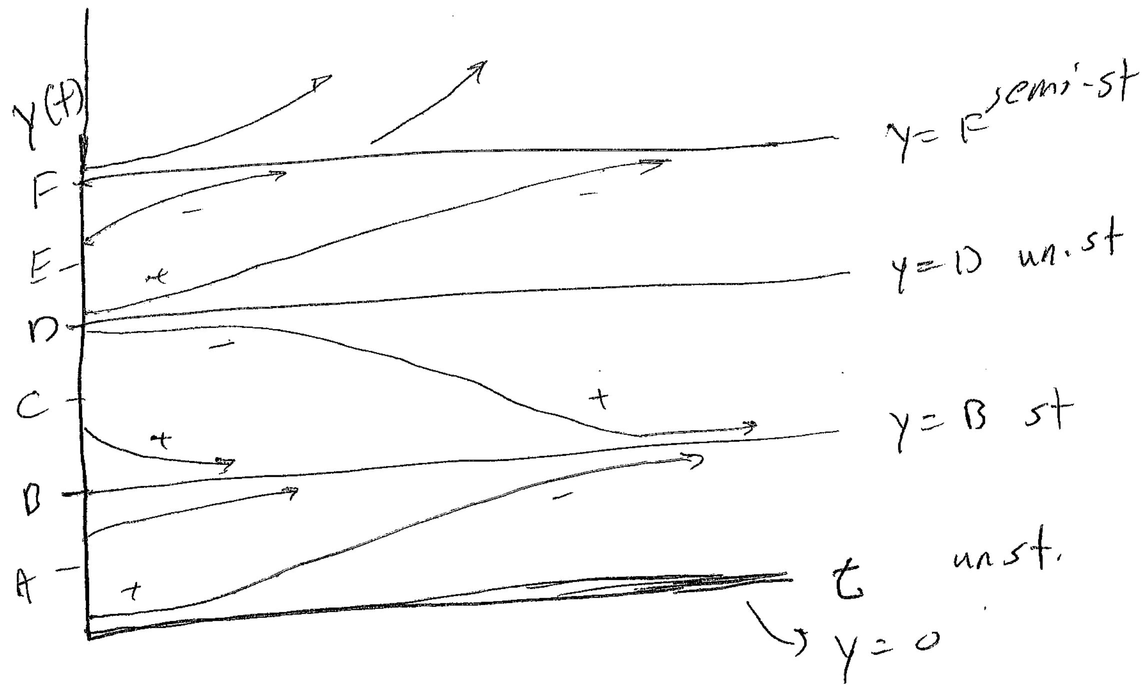
B





6. [10 points] Suppose $y'(t) = F(y(t))$, where the graph of $F(y)$ is given below. Carefully draw several solution curves for this equation. What are the equilibrium solutions? What are their stability types? Describe the initial concavity of the solution curves. Assume $y(t)$ and t are non-negative.





7. [20 points] A boat weighing 480 lbs is pushed forward by a 10 lb force (from its motor). It starts from rest. The water resistance force is twice the velocity.

a. Find a differential equation for the velocity of the boat.

b. Solve it expressing velocity v (ft/sec) and a function of time t seconds.

c. Find the limit of $v(t)$ as $t \to \infty$.

$$F = ma = mv'$$

$$F = 10 - 2v$$

$$mv' = 10 - 2v \qquad\qquad m = \frac{480}{32} = \frac{60}{4} = \frac{30}{2} = 15.$$

$$\frac{dv}{dt} = \frac{10}{m} - \frac{2}{m}v$$

$$v' + \frac{2}{m}v = \frac{10}{m} \qquad \text{Linear}$$

$$v' + \frac{2}{15}v = \frac{10}{15} = \frac{2}{3}$$

$$\mu = e^{\frac{2}{15}t}$$

$$\left(e^{\frac{2}{15}t}v\right)' = \frac{2}{3}e^{\frac{2}{15}t}$$

$$e^{\frac{2}{15}t}v = \underbrace{\frac{2}{3}\cdot\frac{15}{2}}_{=5}e^{\frac{2}{15}t} + C$$

$$v = 5 + Ce^{-\frac{2}{15}t}.$$

$$v(0) = 0 \Rightarrow C = -5.$$

$$v = 5 - 5e^{-\frac{2}{15}t}.$$

$$\lim_{t\to\infty} v(t) = 5.$$

8. [10 BONUS points] In this problem you are to solve

$$\frac{dy}{dx} = \frac{x}{x^2y + y^3}. \qquad (*)$$

Non of our methods will work. Instead we let $u = x^2$ and the use fact that $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ to convert $(*)$ into an equation involving only u and y. Then use the fact that

$$\frac{du}{dy} = \frac{1}{\frac{dy}{du}}$$

to rewrite the equation with u a function of y. It will be linear in u' and u.

You may use $\int y^3 e^{-y^2}\,dy = \frac{-1}{2}(1+y^2)e^{-y^2} + C$.

$u = x^2$

$x = u^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du} 2x = 2u^{\frac{1}{2}}\frac{dy}{du}.$

$2\cancel{u^{\frac{1}{2}}}\frac{dy}{du} = \frac{\cancel{u^{\frac{1}{2}}}}{uy+y^3}$

$\cancel{2(uy+y^3)\frac{dy}{du} = 1}$ does not work

$\frac{1}{2}\frac{du}{dy} = uy + y^3$ linear in u' & u.

$u' - 2yu = y^3$

$\mu = e^{\int -2y\,dy} = e^{-y^2}$

$e^{-y^2}u' - 2ye^{-y^2} = y^3 e^{-y^2}$

$(e^{-y^2}u)' = y^3 e^{-y^2}$

$e^{-y^2}u = \int y^3 e^{-y^2}\,dy = -\frac{1}{2}(1+y^2)e^{-y^2} + C$

$u = \frac{1+y^2}{2} + Ce^{y^2}$

x^2

$x = \pm\sqrt{\frac{1+y^2}{2} + Ce^{y^2}}$