Homework Set 8

"5.2"

1.a. $y'' + xy' - y = 0$. $x_0 = 0$. Let $y = \sum_{n=0}^{\infty} a_n x^n$.

Then $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$. $xy' = \sum_{n=0}^{\infty} n a_n x^n$. $y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \text{ (index shift!)}.$$

Substituting into the given diff eq gives

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0,$$

or $$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n - a_n\right] x^n = 0.$$

Hence $(n+2)(n+1) a_{n+2} + (n-1) a_n = 0$ for $n = 0, 1, 2, 3, \ldots$

Thus, $a_{n+2} = -\dfrac{(n-1) a_n}{(n+2)(n+1)}$ or $a_n = -\dfrac{(n-3) a_{n-2}}{n(n-1)}$,

$n = 2, 3, \ldots$

1.b. $\quad xy'' + xy' + 3y = 0 \qquad x_0 = 2. \qquad$ Let $y = \sum_{n=0}^{\infty} a_n (x-2)^n$.

$y' = \sum_{n=0}^{\infty} n a_n (x-2)^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1) a_n (x-2)^{n-2}$. Use $x = [(x-2)+2]$

$$[(x-2)+2]\,y' = \sum_{n=0}^{\infty} n a_n (x-2)^n + \sum_{n=0}^{\infty} 2 n a_n (x-2)^{n-1}$$

$$= \quad '' \quad + \sum_{n=0}^{\infty} 2(n+1) a_{n+1} (x-2)^n$$

$$[(x-2)+2]\,y'' = \sum_{n=0}^{\infty} n(n-1) a_n (x-2)^{n-1} + \sum_{n=0}^{\infty} 2n(n-1) a_n (x-2)^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+1) n\, a_{n+1} (x-2)^n + \sum_{n=0}^{\infty} 2(n+2)(n+1)\, a_{n+2} (x-2)^n$$

Thus,

$$\sum_{n=0}^{\infty} \left[(n+1)n\, a_{n+1} + 2(n+2)(n+1)\, a_{n+2} + n a_n + 2(n+1) a_{n+1} + 3 a_n \right] (x-2)^n = 0$$

$$2(n+2)(n+1) a_{n+2} + (n+2)(n+1) a_{n+1} + (n+3) a_n = 0 \qquad n = 0, 1, 2, \ldots$$

$$a_{n+2} = -\frac{(n+2)(n+1) a_{n+1} + (n+3) a_n}{2(n+2)(n+1)} \qquad n = 0, 1, 2, \cdots$$

or $$a_n = -\frac{n(n-1) a_{n-1} + (n+1) a_{n-2}}{2n(n-1)} \qquad n = 2, 3, 4, \cdots$$

1.c. $y'' + x^3 y' + 3y = 0 \qquad x_0 = 0.$ Let $y = \sum_{n=0}^{\infty} a_n x^n$.

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}. \qquad x^3 y' = \sum_{n=0}^{\infty} n a_n x^{n+2} = \sum_{n=2}^{\infty} (n-2) a_{n-2} x^n$$

← This will be tricky.

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

I will write out the first few terms of each sum.

$$
\begin{array}{llllll}
y'' = & 2\cdot 1 a_2 & + 3\cdot 2 a_3 x & + 4\cdot 3 a_4 x^2 + & 5\cdot 4 a_5 x^3 + 6\cdot 5 a_6 x^4 + & (n+2)(n+1) a_{n+2} x^n + \\
x^3 y' = & 0 & 0 & 0 a_0 x^2 & 1\cdot a_1 x^3 + 2 a_2 x^4 + \cdots & (n-2) a_{n-2} x^n + \cdots \\
3y = & 3a_0 + & 3a_1 x & + 3a_2 x^2 & + 3a_3 x^3 + 3a_4 x^4 + \cdots & 3a_n x^n + \cdots \\
\hline
0 & & & & &
\end{array}
$$

$2a_2 + 3a_0 = 0 \Rightarrow a_2 = \frac{-3a_0}{2}$

$6a_3 + 3a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{2}$

$12a_4 + 3a_2 = 0 \Rightarrow a_4 = -\frac{a_2}{4}$

But from now on

$$a_{n+2} = -\frac{(n-2)a_{n-2} + 3a_n}{(n+2)(n+1)}, \quad n = 3, 4, \ldots$$

"5.2"

2. a. $W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

If y_1 and y_2 where L.D. $W = 0$ always.
Thus y_1 and y_2 are L.I.

b. $a_0 = 1$, $a_1 = 0$.

$$a_2 = \frac{(2-3)a_0}{2(1)} = -\frac{a_0}{2} = -\frac{1}{2}$$

$$a_3 = \frac{(3-3)a_1}{3\cdot 2} = \frac{0\cdot 0}{6} = 0$$

$$a_4 = \frac{(4-3)a_2}{4\cdot 3} = \frac{a_2}{12} = \frac{-1}{24}$$

$$a_5 = \frac{(5-3)a_3}{5\cdot 4} = 0$$ (all odd terms are zero)

$$a_6 = \frac{(6-3)a_4}{6\cdot 5} = \frac{-1}{240}$$

$a_7 = 0$

$$a_8 = \frac{(8-3)a_6}{8\cdot 7} = \frac{-1}{2688}$$

$a_9 = 0$

$$y_1(x) \approx 1 - \frac{x^2}{2} - \frac{x^4}{24} - \frac{x^6}{240} - \frac{x^8}{2688}$$

$a_0 = 0$ $a_1 = 1$.

$$a_2 = \frac{(2-3)a_0}{2(1)} = 0 \qquad a_4 = \frac{(4-3)a_2}{4\cdot 3} = 0$$ all even terms are zero.

$$a_3 = \frac{(3-3)a_1}{3\cdot 2} = 0$$ all other odd terms $(n \neq 1)$ are zero!

$y_2(x) = x$, exactly. Plug it into $y'' + xy' - y = 0$ and check!

2.c. See link.

"5.3" 1.a. $y'' = \sin x\, y' + 3x^2 y$. $y(0)=1$, $y'(0)=2$.

$a_0 = 1 \quad a_1 = 2$

$a_n = \dfrac{y^{(n)}(0)}{n!}$

$y''(0) = 0 \cdot 2 + 3 \cdot 0^2 \cdot 1 = 0$

$a_2 = \frac{0}{2} = 0$

$y'''(x) = \cos x\, y' + \sin x\, y'' + 6x\, y + 3x^2 y'$

$y'''(0) = 1 \cdot y'(0) + 0 \cdot y''(0) + 6 \cdot 0 \cdot y(0) + 3 \cdot 0^2\, y'(0)$

$= 1 \cdot 2 = 2. \qquad a_3 = \frac{2}{3!} = \cancel{\frac{1}{6}}\ \frac{1}{3}$

$y''''(x) = -\sin x\, y' + \cos x\, y'' + \cos x\, y'' + \sin x\, y''' +$
$6y + 6xy' + 6xy' + 3x^2 y''$.

$y''''(0) = 0 + 1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + 6 \cdot 1 + 0 + 0 + 0 = 6$

$a_4 = \frac{6}{4!} = \frac{1}{4}$

$y(x) \approx 1 + 2x + 0x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$

1.b. $y'' = 2e^{-x}y' - xe^{-x}y$ $\quad y(0)=2 \quad y'(0)=0.$

$a_0 = 2 \quad a_1 = 0$

$y''(0) = 2e^0 \cdot 0 - 0 \cdot e^0 \cdot 2 = 0$ $\qquad a_n = \frac{y^{(n)}(0)}{n!}$

$a_2 = \frac{0}{2!} = 0$

$y'''(x) = -2e^{-x}y' + 2e^{-x}y'' - e^{-x}y + xe^{-x}y - xe^{-x}y'$

$y'''(0) = -0 + 2\cdot 0 - e^0 \cdot 2 + 0 - 0 = -2$

$a_3 = \frac{-2}{3!} = \frac{-1}{3}$

$y''''(x) = 2e^{-x}y' - 2e^{-x}y'' \cancel{-2e^{-x}y''} \cancel{+2e^{-x}y''}$

$+ e^{-x}y - e^{-x}y' + e^{-x}y - xe^{-x}y + xe^{-x}y'$

$- e^{-x}y' + xe^{-x}y' - xe^{-x}y''.$

$y''''(0) = 0 - 2\cdot 0 + e^0 \cdot 2 - e^0 \cdot 0 + e^0 2 + 0 + 0 - 0 + 0 - 0$

$= 4$

$a_4 = \frac{-4}{4!} = \frac{-1}{3!} = \frac{-1}{6}$

$y(x) \approx 2 + 0x + 0x^2 - \frac{x^3}{3} - \frac{x^4}{6}$

c. $y'' = x^2 y' + 3y$ $\qquad y(3) = 1 \quad y'(3) = 1.$

$$y = \sum_{n=1}^{\infty} a_n (x-3)^n$$

$$a_n = \frac{y^{[n]}(3)}{n!}$$

$a_0 = 1 \quad a_1 = 1.$

$$y''(3) = 9 \cdot y'(3) + 3 \cdot y(3)$$

$$= 9 + 3 = 12$$

$$a_2 = \frac{12}{2!} = 6.$$

$$y'''(x) = 2xy' + x^2 y'' + 3y'$$

$$y'''(3) = \cancel{0 + 0 + 3 \cdot 1 = 3} = 2 \cdot 3 \cdot 1 + 9 \cdot 6 + 3 \cdot 1$$

$$= 63$$

$$\cancel{a_3 = \frac{3}{3!} = \frac{1}{2}} \qquad a_3 = \frac{63}{3!} = 10.5$$

$$y''''(x) = 2y' + \underline{2xy'' + 2xy''} + x^2 y''' + 3y''$$

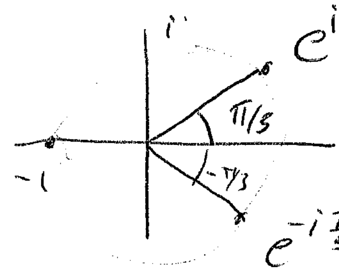
$$y''''(3) = 2 \cdot 1 + \overset{2x}{2} \cdot 3 \cdot 6 + 9 \cdot 63 + 3 \cdot 6 = 659$$

$$a_4 = \frac{659}{4!} = 27.458\overline{3}.$$

5.3

2a $\quad y'' + \frac{4x}{1+x^3}y' + \frac{1}{1+x^3}y = 0 \qquad x_0 = 3.$

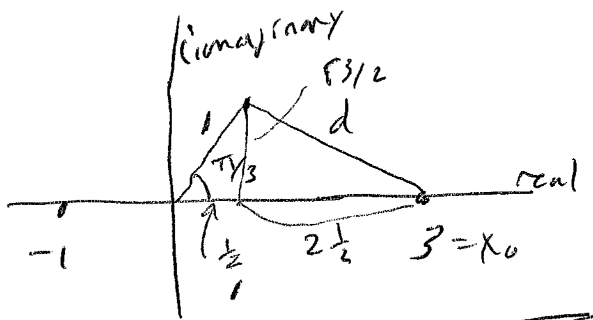
$1 + x^3 = 0 \iff x^3 = -1$



$e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$= \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$e^{-i\frac{\pi}{3}}$



why? $\left(e^{i\frac{\pi}{3}}\right)^3 = e^{i\pi} = \cos\pi + i\sin\pi$

$= -1$

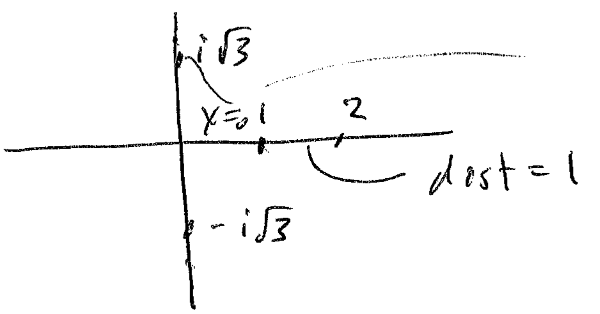
$\left(e^{-i\frac{\pi}{3}}\right)^3 = e^{-i\pi} = \cos(-\pi) + i\sin(-\pi)$

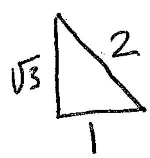
$= -1$

$d = \sqrt{\left(2\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{25}{4} + \frac{3}{4}} = \frac{\sqrt{28}}{2}$ ← radius will be at least this.

b. $\quad y'' + \frac{1}{x-2}y' + \frac{1}{x^2+3}y = 0 \qquad x_0 = 1$





min dist is 1.

radius is at least 1.