Name: ______________________ Section time: ______________________

NONGRAPHING CALCULATORS ALLOWED

(1) [10 points each] Find the general solution to each of the following. Solve for the dependent variable if possible.

(a) $y\sec^2 x + (\tan x)y' = 0$.

Solution. This is exact and seperable (in fact all separable examples are exact - why?). Let $M = y\sec^2 x$ and $N = \tan x$. Then $M_y = \sec^2 x$ and $N_x = \sec^2 x$. It is easy to get $\psi = y\tan x$. So, the general solution is $y\tan x = C$ or $y = C\cot x$.

To use the separable method start with $\frac{y'}{y} = \frac{-\sec^2 x}{\tan x}$. Thus we need

$$\int \frac{1}{y}\,dy = -\int \frac{\sec^2 x}{\tan x}\,dx$$

For the righthnad side let $u = \tan x$. Then

$$\ln|y| = -\ln|\tan x| + C = \ln|\cot x| + C.$$

And,

$$y = \pm\exp(\ln|\cot x| + C) = \pm e^C \cdot |\cot x|.$$

So,

$$y = C\cot X$$

as before. (We can choose C so that we don't need the absolute value of the cotangent since we know the solution has to be smooth, that is differentiable.) □

(b) $y' + y^2\ln x = 0$.

Solution. This is separable. We get

$$\int y^{-2}\,dy = -\int \ln x\,dx = -(x\ln x - x + C).$$

Thus,

$$-1/y = -(x\ln x - x + C).$$

So,

$$y = \frac{1}{x\ln x - x + C}.$$

Notice it is also a Bernoulli type equation with $n = 2$. □

(c) $y' = \dfrac{xy}{x^2 + y^2}$.

Solution. This one is homogeneous. Let $v = y/x$. Then $xv = y$ so $y' = v + xv'$. If we multiply the righthand side x^{-2}/x^{-2} the equation becomes

$$v + xv' = \frac{v}{1 + v^2}$$

. So,

$$x\frac{dv}{dx} = \frac{v}{1+v^2} - v = \frac{v}{1+v^2} - \frac{v(1+v^2)}{1+v^2} = \frac{-v^3}{1+v^2}.$$

Hence,

$$\frac{1+v^2}{v^3}dv = \frac{-1}{x}dx.$$

This becomes

$$\int v^{-3} + v^{-1}dv = -\ln|x| + C \text{ or } -(1/2)v^{-2} + \ln|v| = -\ln|x| + C.$$

Recall $v = y/x$. Thus, using $\ln|y/x| = \ln|y| - \ln|x|$, we get

$$-\frac{x^2}{2y^2} + \ln|y| - \ln|x| = -\ln|x| + C.$$

Now,

$$-x^2 = -2y^2\ln|y| + Cy^2 = y^2(C - \ln y^2).$$

Finally,

$$x = \pm\sqrt{y^2(\ln(y^2) - C)}.$$

□

(d) $t^2y' + 2ty = y^3$ where $t > 0$.

Solution. This is Bernoulli and I don't think any other method we have will work. Let $v = y^{1-3}$. So, $y = v^{-1/2}$. Thus, $y' = (-1/2)v^{-3/2}v'$ and the equation becomes

$$t^2(-1/2)v^{-3/2}v' + 2tv^{-1/2} = v^{-3/2}.$$

Thus,

$$-(t^2/2)v' + 2tv = 1 \text{ or } v' - (4/t)v = -2t^{-2}.$$

Now it is linear. You can check that $\mu = t^{-4}$ is the integrating factor. This gives

$$t^{-4}v' - 4t^{-5}v = -2t^{-6}.$$

Thus,

$$(t^{-4}v)' = -2t^{-6}$$

and so,

$$t^{-4}v = (2/5)t^{-5} + C.$$

Hence,

$$v = 2/(5t) + Ct^4.$$

But $v = 1/y^2$ so

$$y = \pm\sqrt{\frac{1}{\frac{2}{5t} + Ct^4}}.$$

□

(2) [10+5+5+5 points] (a) Find the particular continuous solution to

$$y' + p(t)y = 0 \text{ where } y(0) = 1$$

and

$$p(t) = \begin{cases} 2 & 0 \le t \le 1 \\ -1 & t > 1. \end{cases}$$

(b) Sketch the solution.
(c) What is the minimum value of $y(t)$ and when does it occur?
(d) At what time t will $y(t) = 10$?

Solution. (a) For $t \in [0, 1]$ we have $y' = -2y$. Hence the general solution is $y = Ce^{-2t}$. Since $y(0) = 1$ we get $C = 1$. Let $y_1(t) = e^{-2t}$.

For $t \in [0, 1]$ we have $y' = y$. Hence the general solution is $y = Ce^t$. Since $y(1) = y_1(1) = e^{-2}$ we get $C = e^-3$. Let $y_2(t) = e^{t-3}$.

Thus our solution is

$$y(t) = \begin{cases} e^{-2t} & 0 \le t \le 1 \\ e^{t-3} & t > 1. \end{cases}$$

(b) This is easy to graph.
(c) The minimum value of e^{-2} occurs when $t = 1$.
(d) It is clear that $y(t)$ will reach 10 for some time $t > 1$. So we need to solve $e^{t-3} = 10$. This gives

$$t = 3 + \ln 10 \approx 5.30 \text{ time units.}$$

□

(3) [15 points] A tank starts with 100 liters of fresh water.

It can hold 300 liters before it will overflow.

Water that has 3 grams salt per liter is pumped in at 3 liters per minute.

The well mixed solution flows out of the tank at 2 liters per minute.

How many grams of salt will be in the solution in this tank when it just begins to overflow?

Solution.

$$\frac{dQ}{dt} = \text{ rate of salt in } - \text{ rate of salt out}$$

$$\frac{dQ}{dt} = 3 \times 3 - 2 \times \frac{Q}{V}$$

But $V(t) = 100 + t$. So we get

$$Q' + \frac{2Q}{100 + t} = 9.$$

This is linear. Let $\mu = e^{\int \frac{2}{100+t}\,dt} = (100 + t)^2$. So now we have

$$(100 + t)^2 Q' + 2(100 + t)Q = 9(100 + t)^2.$$

Using the Product Rule backwards gives us

$$((100 + t)^2 Q)' = 9(100 + t)^2.$$

Integrating gives

$$(100 + t)^2 Q = 3(100 + t)^3 + C.$$

Hence

$$Q(t) = 3(100 + t) + \frac{C}{(100 + t)^2}.$$

At $t = 0$ we know $Q(0) = 0$. This determines C. We get $C = -3,000,000$. Overflow happens at $t = 200$. Thus

$$Q(200) = 866\frac{2}{3} \text{ grams of salt.}$$

□